Enumerating conjugacy classes of graphical groups

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Graphical groups (over the integers)

Let Γ be a (finite, simple, undirected) graph with vertices $v_1, \ldots, v_n$.

Let $\mathcal{I}$ consist of those (i, j) with $i < j$ and such that $v_i \sim v_j$.

Definition

The **graphical group** $\mathbf{G}_\Gamma(\mathbf{Z})$ is defined as follows:

- Generators: $x_1, \ldots, x_n$ and z_{ij} for $(i, j) \in \mathcal{I}$.
- Relations:
  - For $i < j$,
    $$[x_i, x_j] = \begin{cases} z_{ij}, & \text{if } (i,j) \in \mathcal{I}, \\ 1, & \text{otherwise.} \end{cases}$$
  - Each z_{ij} is central.

Example

- $\mathbf{G}_{K_2}(\mathbf{Z}) \approx U_3(\mathbf{Z})$ (discrete Heisenberg group)
- $\mathbf{G}_{K_d}(\mathbf{Z}) \approx F_{2,d}(\mathbf{Z})$ is the free nilpotent group of class $\leqslant 2$ on d generators.
- $\mathbf{G}_{P_d}(\mathbf{Z}) \approx U_{d+1}(\mathbf{Z})/\gamma_3(U_{d+1}(\mathbf{Z}))$

Graphical groups (over rings)

Let R be a commutative ring.

Definition

The **graphical group** $\mathbf{G}_\Gamma(R)$ is defined as follows:

- Generators: $x_1^r, \ldots, x_n^r$ and z_{ij}^r for $r \in R$ and $(i,j) \in \mathcal{I}$.
- Relations:
  - $x_i^r x_i^s = x_i^{r+s}$ and $z_{ij}^r z_{ij}^s = z_{ij}^{r+s}$.
  - For $i < j$,
  $$[x_i^r, x_j^s] = \begin{cases} z_{ij}^{rs}, & \text{if } (i,j) \in \mathcal{I}, \\ 1, & \text{otherwise.} \end{cases}$$
  - Each z_{ij}^r is central.

Question

Let R be a “friendly” finite ring (e.g. $R = \mathbf{F}_q$).
How do group-theoretic properties of $\mathbf{G}_\Gamma(R)$ depend on R?

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Let $\mathcal{O}$ be a local PID with maximal ideal $\mathfrak{m}$. Suppose that $q := |\mathcal{O}/\mathfrak{m}|$ is *finite*.

Example

- Finite fields: $\mathbf{F}_q$.
- Univariate power series rings over finite fields: $\mathbf{F}_q[\![z]\!]$.
- Rings of integers of finite extensions of $\mathbf{Q}_p$ (p prime).

In fact, these are the only complete examples. We'll assume that $\mathcal{O}$ is *not* a field.

Let

$$Z^{cc}_{\mathbf{G}_\Gamma,\mathcal{O}}(T) = \sum_{k=0}^{\infty} k(\mathbf{G}_\Gamma(\mathcal{O}/\mathfrak{m}^k))T^k.$$

Combinatorics in, combinatorics out

Theorem ("Uniformity Theorem"; R. & Voll 2024, 2025^+)

Let Γ *have* m *edges. There exists* $W_\Gamma(X,T) \in \mathbf{Q}(X,T)$ *such that for each* $(\mathcal{O},\mathfrak{m})$,

$$Z^{cc}_{\mathbf{G}_\Gamma,\mathcal{O}}(T) = W_\Gamma(q, q^m T).$$

Corollary

Given Γ, *the number* $k(\mathbf{G}_\Gamma(\mathbf{F}_q))$ *is a polynomial in* q.

We have two algorithms for computing $W_\Gamma(X,T)$. Both are based on *toric geometry*. No explicit "combinatorial formula" is known for $W_\Gamma(X,T)$.

Question

What are the effects of natural graph-theoretic operations on $W_\Gamma(X,T)$?

Joins

The **join** $\Gamma_1 \vee \Gamma_2$ of graphs Γ_1 and Γ_2 is obtained from the disjoint union $\Gamma_1 \oplus \Gamma_2$ by joining each vertex of Γ_1 to each vertex of Γ_2.

Fact

$\mathbf{G}_{\Gamma_1 \vee \Gamma_2}(\mathbf{Z}) \approx (\mathbf{G}_{\Gamma_1}(\mathbf{Z}) * \mathbf{G}_{\Gamma_2}(\mathbf{Z}))/\gamma_3$.

Let $n_1, n_2 \geqslant 0$ be given. Write $z_i = X^{-n_i}$. Given $W_1(X,T), W_2(X,T) \in \mathbf{Q}(X,T)$, define

$$(W_1 \underset{n_1 \vee n_2}{\diamond} W_2)(X,T)$$
$$= \frac{z_1 z_2 XT - 1 + W_1(X, z_2 T)(1 - z_2 T)(1 - z_2 XT) + W_2(X, z_1 T)(1 - z_1 T)(1 - z_1 XT)}{(1-T)(1-XT)}$$
$$= \text{harmless} + W_1(X, z_2 T) \cdot \text{harmless} + W_2(X, z_1 T) \cdot \text{harmless}.$$

Theorem (R. & Voll 2025$^+$)

Let Γ_i have n_i vertices. Write $z_i = X^{-n_i}$. Then $W_{\Gamma_1 \vee \Gamma_2}(X,T) = W_{\Gamma_1}(X,T) \underset{n_1 \vee n_2}{\diamond} W_{\Gamma_2}(X,T)$.

For *cographs*, this was already known (R. & Voll 2024).

Disjoint unions

The **Hadamard product** of $F = \sum_{k=0}^{\infty} a_k T^k$ and $G = \sum_{k=0}^{\infty} b_k T^k$ is $F *_T G = \sum_{k=0}^{\infty} a_k b_k T^k$.

Fact

$W_{\Gamma_1 \oplus \Gamma_2} = W_{\Gamma_1} *_T W_{\Gamma_2}$.

Theorem (Carnevale, Moustakas, R. 2025)

$$W_{K_{d_1} \oplus \cdots \oplus K_{d_n}}(X, T) = \frac{\textit{an explicit sum over } B_n = \{\pm 1\} \wr S_n}{(1 - T)(1 - XT) \cdots (1 - X^n T)}$$

Corollary (Carnevale, Moustakas, R. 2025)

Fix $n \geqslant 1$. *There is an explicit rational function* $W_n(X, Y_1, \dots, Y_n, T)$ *such that for all* $d_1, \dots, d_n$ *and for each* $(\mathcal{O}, \mathfrak{m})$,

$$Z^{cc}_{F_{2,d_1} \times \cdots \times F_{2,d_n}, \mathcal{O}}\left(q^{-\sum_{i=1}^{n} \binom{d_i}{2}} T\right) = W_n(q, q^{d_1}, \dots, q^{d_n}, T).$$

We're only beginning to understand the combinatorics of Hadamard products of the $W_\Gamma(X, T)$.

Thank you!