# Topological representation zeta functions of unipotent groups

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Inspired by work surrounding Igusa's local zeta function, we introduce topological representation zeta functions of unipotent algebraic groups over number fields. These group-theoretic invariants capture common features of established $p$-adic representation zeta functions associated with pro-$p$ groups derived from unipotent groups. We investigate fundamental properties of the topological zeta functions considered here. We also develop a method for computing them under non-degeneracy assumptions. As an application, among other things, we obtain a complete classification of topological representation zeta functions of unipotent algebraic groups of dimension at most six.

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## 1 Introduction

**Enumerating representations.** Representation zeta functions of groups are Dirichlet series enumerating irreducible finite-dimensional complex representations up to suitable notions of equivalence. For instance, given a group $G$, let $r_n(G)$ be the possibly infinite number of irreducible representations $G \to \mathrm{GL}_n(\mathbf{C})$ counted up to equivalence of representations in the usual sense; if $G$ is a topological group, then we only consider continuous representations. For various interesting classes of groups, it turns out that the numbers $r_n(G)$ are polynomially bounded as a function of $n$. For such groups, we may then consider the representation zeta function $\sum_{n=1}^{\infty} r_n(G)n^{-s}$, where $s$ is a complex variable. A part of the theory of representation growth, the study of such zeta functions is an active area, see [17, 38] for surveys. Among the classes of groups of major interest are arithmetic groups associated with semisimple algebraic groups [18, 20], compact $p$-adic analytic groups [2, 16], as well as nilpotent and unipotent groups [15, 34].

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**Representation zeta functions of nilpotent and unipotent groups: twisting.** For a finitely generated infinite nilpotent group $G$, the number $r_1(G)$ is infinite. Hrushovski and Martin [15] (first version, 2006) studied "twist-isoclasses" of representations. Here, two representations $\varrho_1, \varrho_2 \colon G \to \mathrm{GL}_n(\mathbf{C})$ of an arbitrary group $G$ are **twist-equivalent** if $\varrho_1$ is equivalent to $\alpha \otimes_{\mathbf{C}} \varrho_2$ in the usual sense for a 1-dimensional representation $\alpha \colon G \to \mathbf{C}^\times$; for a topological group $G$, we require $\varrho_1$, $\varrho_2$, and $\alpha$ to be continuous. Twist-equivalence classes of representations are called **twist-isoclasses**. Let $\tilde{r}_n(G)$ denote the number of twist-isoclasses of (continuous) irreducible representations $G \to \mathrm{GL}_n(\mathbf{C})$. If $G$ is finitely generated nilpotent or the pro-$p$ completion of such a group, then the $\tilde{r}_n(G)$ are polynomially bounded and the **representation zeta function** of $G$ is

$$\zeta_G(s) := \sum_{n=1}^{\infty} \tilde{r}_n(G) n^{-s}.$$

Such zeta functions were studied in [36, §3.4] and [15, §8]. Denoting the pro-$p$ completion of a finitely generated nilpotent group $G$ by $\hat{G}_p$, a central theme in both cited sources is the behaviour of $\zeta_{\hat{G}_p}(s)$ under variation of $p$. Extending [36, §3.4], Stasinski and Voll [34] studied representation zeta functions enumerating twist-isoclasses of groups arising from unipotent group schemes. This is essentially the point of view taken here.

**Previously considered types of topological zeta functions.** Classical topological zeta functions are singularity invariants of hypersurfaces. Given $f \in \mathbf{Z}[X_1, \dots, X_n]$, the topological zeta function of $f$ was first defined by Denef and Loeser [7] by means of a limit "$p \to 1$" applied to Igusa's $p$-adic zeta function associated with $f$. Later, the topological zeta function of $f$ was reinterpreted within the framework of motivic integration [8]. Not only do topological zeta functions retain crucial features of their $p$-adic ancestors (see [7, Thm 2.2]), they have also been found to be more amenable to both theoretical investigations and explicit computations, see e.g. [19, 24] for some recent developments.

Using connections between subgroup and subring zeta functions and $p$-adic integration going back to [14], topological subalgebra and subgroup zeta functions of Lie algebras and nilpotent groups, respectively, were introduced by du Sautoy and Loeser [11]. In [28, 29], the author developed systematic means of computing these zeta functions in favourable situations. Substantial evidence indicates that they possess remarkable properties which are not present in the hypersurface case, see [28, §8].

**Topological representation zeta functions of unipotent groups.** The purpose of the present article is to initiate the study of topological representation zeta functions and to establish them as interesting group-theoretic invariants that deserve further attention.

First, after recalling some key results from [34] in §2, given a unipotent algebraic group $\mathbf{G}$ over a number field $k$, we define the topological representation zeta function $\zeta_{\mathbf{G},\mathrm{top}}(s) \in \mathbf{Q}(s)$ of $\mathbf{G}$ in §3 using $p$-adic formulae from [34, 36] and ideas from [7]. For a brief and informal sketch, suppose that $k = \mathbf{Q}$ and choose an arbitrary $\mathbf{Z}$-form $\mathsf{G}$ of $\mathbf{G}$ as an affine group scheme. Then for sufficiently large primes $p$, each $\mathsf{G}(\mathbf{Z}_p)$ is a pro-$p$ group whose representation zeta function $\zeta_{\mathsf{G}(\mathbf{Z}_p)}(s)$ enumerating twist-isoclasses is defined as above, where $\mathbf{Z}_p$ denotes the $p$-adic integers. Informally, we then define $\zeta_{\mathbf{G},\mathrm{top}}(s)$ to be the constant term of $\zeta_{\mathsf{G}(\mathbf{Z}_p)}(s)$ as a series in $p-1$. As an example, a $\mathbf{Z}$-form of the Heisenberg group, $\mathbf{H}$ say, over $\mathbf{Q}$ is given by

$$\mathsf{H}(R) := \begin{bmatrix} 1 & R & R \\ 0 & 1 & R \\ 0 & 0 & 1 \end{bmatrix} \leqslant \mathrm{GL}_3(R)$$

for commutative rings $R$. It is well-known that

$$\zeta_{\mathsf{H}(\mathbf{Z}_p)}(s) = \frac{1-p^{-s}}{1-p^{1-s}}$$

for all primes $p$. By formally expanding $p^z = (1+(p-1))^z$ in $p-1$ using the binomial series, we obtain $\zeta_{\mathsf{H}(\mathbf{Z}_p)}(s) = \frac{s}{s-1} + \mathcal{O}(p-1)$ whence $\zeta_{\mathbf{H},\mathrm{top}}(s) = \frac{s}{s-1}$.

Having rigorously defined topological representation zeta functions, we establish some of their basic properties in §4. Perhaps most interestingly, we find that they always have degree zero in $s$. In contrast, the degrees of topological zeta functions of polynomials can vary wildly, and the degrees of topological subalgebra zeta functions are only understood conjecturally, see [28, Conj. I]. Along the way, we also briefly consider topological representation zeta functions attached to perfect Lie algebras in the spirit of [2].

Computations (largely ad hoc) of $p$-adic representation zeta functions of various nilpotent groups can be found in the theses of Ezzat [12] and Snocken [32]. Being derived from $p$-adic zeta functions, we expect topological representation zeta functions to be more easily computable than the former. Indeed, building on the author's previous work [28, 29], in §5, we develop a method for directly computing topological representation zeta functions at least under additional hypotheses. With this method at our disposal, in §6, we provide numerous examples of topological representation zeta functions, including a complete classification in dimension six (see Table 1, p. 26). As a result of our work, our knowledge of specific examples of topological representation zeta functions associated with unipotent groups far exceeds the $p$-adic case. Finally, in §7, we discuss open questions that arose from remarkable patterns exhibited by the known examples of topological representation zeta functions.

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## 2 Local representation zeta functions

We explain how unipotent algebraic groups give rise to pro-$p$ groups. We then recall a result from [34, 36] on the shapes of the representation zeta functions of these groups.

## 2.1 Basic facts on unipotent groups over number fields

**Unipotent algebraic groups.** The following is folklore, see [5, Ch. IV, §2] and [22, Ch. XV]. Let $F$ be a field. By an $F$**-group**, we mean an algebraic group over $F$. As a non-intrinsic characterisation of unipotence, we say that an $F$-group is **unipotent** if for some $n \geqslant 1$, it is isomorphic to an algebraic subgroup of the $F$-group of upper unitriangular $n \times n$-matrices for some $n$. If $F$ is perfect, then a connected linear $F$-group is unipotent if and only if it admits a central series of closed algebraic subgroups whose non-trivial factors are $F$-isomorphic to the additive group.

Let $F$ have characteristic zero. Given a finite-dimensional nilpotent Lie $F$-algebra $\mathfrak{g}$, the Baker-Campbell-Hausdorff series associated with $\mathfrak{g}$ is a polynomial. Using it, we may endow $\mathbf{G}(A) := \mathfrak{g} \otimes_F A$ with a group structure for each commutative $F$-algebra $A$. The functor $A \mapsto \mathbf{G}(A)$ is represented by a unipotent $F$-group which we again denote by $\mathbf{G}$. The construction of $\mathbf{G}$ from $\mathfrak{g}$ is functorial and $\mathfrak{g} \mapsto \mathbf{G}$ furnishes an equivalence between the categories of finite-dimensional nilpotent Lie $F$-algebras and unipotent $F$-groups; a quasi-inverse is given by the usual functor taking an algebraic group to its Lie algebra.

**Notation for number fields and their places.** Throughout this article, we let $k$ denote a number field with ring of integers $\mathfrak{o}$. We let $\mathcal{V}_k$ denote the set of all non-Archimedean places of $k$. Given $v \in \mathcal{V}_k$, we let $k_v$ denote the $v$-adic completion of $k$. We further let $\mathfrak{o}_v$ and $\mathfrak{K}_v$ denote the valuation ring and residue field of $k_v$, respectively, and write $q_v = |\mathfrak{K}_v|$. Let $\mathfrak{p}_v \in \mathrm{Spec}(\mathfrak{o})$ be the prime ideal corresponding to $v$ and let $p_v$ be the rational prime contained in $\mathfrak{p}_v$. For a $p$-adic field $K$, i.e. a finite extension of the field $\mathbf{Q}_p$ of $p$-adic numbers, let $\mathfrak{O}_K$ denote the valuation ring of $K$, let $\mathfrak{P}_K$ be the maximal ideal of $\mathfrak{O}_K$, and write $q_K = |\mathfrak{O}_K/\mathfrak{P}_K|$. We let $|\cdot|_K$ and $\|\cdot\|_K$ denote the usual $\mathfrak{P}_K$-adic absolute value and maximum norm, respectively; in particular, $|\pi|_K = q_K^{-1}$ for $\pi \in \mathfrak{P}_K \setminus \mathfrak{P}_K^2$.

**Integral forms and Lie groups.** Let $\mathbf{G}$ be a linear algebraic group over $k$. By an $\mathfrak{o}$**-form** of $\mathbf{G}$, we mean an affine group scheme $\mathsf{G}$ of finite type over $\mathfrak{o}$ with $\mathsf{G} \otimes_{\mathfrak{o}} k \approx_k \mathbf{G}$. For a commutative ring $R$, write $\mathrm{GL}_{n,R} = \mathrm{GL}_n \otimes_{\mathbf{Z}} R$. Every $k$-isomorphism from $\mathbf{G}$ onto an algebraic subgroup of some $\mathrm{GL}_{n,k}$ provides us with an $\mathfrak{o}$-form of $\mathbf{G}$ as a closed subgroup scheme of $\mathrm{GL}_{n,\mathfrak{o}}$, see [13, §1]. If $\mathsf{G}$ is an arbitrary $\mathfrak{o}$-form of $\mathbf{G}$, there exists $N \in \mathbf{N}$ such that $\mathsf{G} \otimes_{\mathfrak{o}} \mathfrak{o}[1/N]$ embeds as a closed subgroup scheme into $\mathrm{GL}_{n,\mathfrak{o}[1/N]}$. Define $S = \{v \in \mathcal{V}_k : p_v \mid N\}$. If $v \in \mathcal{V}_k \setminus S$, then we may regard each $\mathsf{G}(\mathfrak{o}_v) \leqslant \mathrm{GL}_n(\mathfrak{o}_v)$ as a compact $p_v$-adic Lie group. If $\mathbf{G}$ is unipotent, then for almost all $v \in \mathcal{V}_k \setminus S$, the group $\mathsf{G}(\mathfrak{o}_v)$ is a torsion-free, nilpotent, finitely generated pro-$p_v$ group. While the family $(\mathsf{G}(\mathfrak{o}_v))_{v \in \mathcal{V}_k \setminus S}$ of topological groups depends on the choices made, it does so in a mild way. Namely, if $\tilde{\mathsf{G}}$ is another $\mathfrak{o}$-form of $\mathbf{G}$, then there exists a finite set $\tilde{S} \subset \mathcal{V}_k$ such that $\mathsf{G}(\mathfrak{o}_v) \approx \tilde{\mathsf{G}}(\mathfrak{o}_v)$ as topological groups for $v \in \mathcal{V}_k \setminus \tilde{S}$; we may even assume that $\mathsf{G}(\mathfrak{O}_K) \approx \tilde{\mathsf{G}}(\mathfrak{O}_K)$ for finite extensions $K/k_v$.

## 2.2 Representation zeta functions associated with unipotent groups

We recall the setting and statement of [34, Thm A] in a form convenient for our purposes. First, we record the following elementary fact.

**Lemma 2.1** (Cf. [34, Lemma 2.1]). *Let $G$ be a finitely generated nilpotent pro-$p$ group. Then $\tilde{r}_n(G) < \infty$ for all $n \in \mathbf{N}$ and $\tilde{r}_n(G) = \mathcal{O}(n^\alpha)$ for some real number $\alpha \geqslant 0$.*

*Proof.* Since $G$ is $p$-adic analytic, it has polynomial subgroup growth [9, Thm 3.19]. The proof of [34, Lemma 2.1] for the case of $\mathcal{T}$-groups now carries over to the present case. ♦

In particular, the definition of $\zeta_G(s)$ given in the introduction makes perfect sense if $G$ is any finitely generated nilpotent pro-$p$ group. Recall that a finite group is monomial if and only if each of its irreducible complex representations is induced from a 1-dimensional representation. Since finite $p$-groups are monomial (cf. [27, 8.4.9]), we see that $\tilde{r}_n(G) \neq 0$ only if $n$ is a power of $p$. By §2.1, if $\mathsf{G}$ is an $\mathfrak{o}$-form of a unipotent $k$-group, then Lemma 2.1 applies to almost all of the groups $\mathsf{G}(\mathfrak{o}_v)$ for $v \in \mathcal{V}_k$. The following result explains the behaviour of $\zeta_{\mathsf{G}(\mathfrak{o}_v)}(s)$ under variation of $v$.

**Theorem 2.2** (Pf of [34, Thm A]; cf. [36, Thm D]). *Let $\mathsf{G}$ be an $\mathfrak{o}$-form of a unipotent $k$-group. Then there are separated $\mathfrak{o}$-schemes $V_1, \dots, V_r$ of finite type, rational functions $W_1, \dots, W_r \in \mathbf{Q}(X, Y)$, and a finite set $S \subset \mathcal{V}_k$ such that if $v \in \mathcal{V}_k \setminus S$ and $K$ is any finite extension of $k_v$, then $\zeta_{\mathsf{G}(\mathfrak{O}_K)}(s) = \sum\limits_{i=1}^{r} \#V_i(\mathfrak{O}_K/\mathfrak{P}_K) \cdot W_i(q_K, q_K^{-s})$—in particular, $\zeta_{\mathsf{G}(\mathfrak{O}_K)}(s)$ is rational in $q_K^{-s}$ over $\mathbf{Q}$ and admits meromorphic continuation to all of $\mathbf{C}$.*

**Remark 2.3.** Let $\mathbf{\mathfrak{g}}$ be a finite-dimensional nilpotent Lie $k$-algebra. Choose an $\mathfrak{o}$-form $\mathfrak{g}$ of $\mathbf{\mathfrak{g}}$ which is free as an $\mathfrak{o}$-module and which satisfies $[\mathfrak{g}, \mathfrak{g}] \subset c!\mathfrak{g}$, where $c$ is the nilpotency class of $\mathbf{\mathfrak{g}}$. Stasinski and Voll [34, §2.1.2] used the following $\mathfrak{o}$-form $\mathsf{G}$ of the unipotent $k$-group corresponding to $\mathbf{\mathfrak{g}}$. Namely, for a commutative $\mathfrak{o}$-algebra $R$, define a group structure on $\mathsf{G}(R) = \mathfrak{g} \otimes_{\mathfrak{o}} R$ using the Baker-Campbell-Hausdorff series, exactly as in §2.1. The resulting group scheme $\mathsf{G}$ over $\mathfrak{o}$ is unipotent in the sense of [34, §2.1.1], i.e. $\mathsf{G}$ is affine, smooth, and its geometric fibres are connected unipotent algebraic groups. For our purposes, we may discard finitely many places of $k$ as needed whence the particular choice of an $\mathfrak{o}$-form is immaterial.

# 3 Topological representation zeta functions

In the preceding section, given a unipotent algebraic group $\mathbf{G}$ over $k$, after choosing an arbitrary $\mathfrak{o}$-form $\mathsf{G}$ of $\mathbf{G}$, we obtained a family $(\mathsf{G}(\mathfrak{o}_v))_{v \in \mathcal{V}_k \setminus S}$ of groups and associated representation zeta functions. In this section, we define the topological representation zeta function of $\mathbf{G}$ by means of a limit "$q_v \to 1$" applied to the zeta functions $\zeta_{\mathsf{G}(\mathfrak{o}_v)}(s)$.

## 3.1 Taking the limit "$q \to 1$"

In the study of local zeta functions, we encounter families of functions of the form $W(q_v, q_v^{-s_1}, \dots, q_v^{-s_l})$, where $W \in \mathbf{Q}(X, Y_1, \dots, Y_l)$ and $v$ runs over (almost all elements of) $\mathcal{V}_k$. In this subsection, which summarises [28, §5.1], we describe conditions on the shape of $W$ that allow us to pass to the limit "$q_v \to 1$", denoted $\lfloor W \rfloor$ below, by taking the constant term of $W(q_v, q_v^{-s_1}, \dots, q_v^{-s_l})$ symbolically expanded as a series in $q_v - 1$.

Given a polynomial $e \in \mathbf{Q}[s_1, \dots, s_l]$, using the binomial series, we formally expand

$$X^e := (1 + (X-1))^e := \sum_{d=0}^{\infty} \binom{e}{d} (X-1)^d \in \mathbf{Q}[s_1, \dots, s_l][\![X-1]\!].$$

The rule $f \mapsto f(X, X^{-s_1}, \dots, X^{-s_l})$ then extends to an embedding of $\mathbf{Q}(X, Y_1, \dots, Y_l)$ into the field $\mathbf{Q}(s_1, \dots, s_l)(\!(X-1)\!)$ of formal Laurent series in $X - 1$ over $\mathbf{Q}(s_1, \dots, s_l)$.

**Definition 3.1.** Let $\mathbf{M}[X, Y_1, \dots, Y_l]$ be the $\mathbf{Q}$-algebra consisting of all those rational functions $W \in \mathbf{Q}(X, Y_1, \dots, Y_l)$ satisfying the following two conditions:

(a) $W$ can be written in the form

$$W = f \cdot \prod_{(a,b) \in \mathbf{Z}^{1+l} \setminus \{0\}} (1 - X^a Y_1^{b_1} \cdots Y_l^{b_l})^{-e(a,b)},$$

where $f \in \mathbf{Q}[X^{\pm 1}, Y_1^{\pm 1}, \dots, Y_l^{\pm 1}]$, $e(a,b) \in \mathbf{N} \cup \{0\}$ and $e(a,b) = 0$ for almost all $(a,b)$.

(b) $W(X, X^{-s_1}, \dots, X^{-s_l}) \in \mathbf{Q}(s_1, \dots, s_l)[\![X-1]\!]$ is a *power series* in $X - 1$ (instead of merely a Laurent series).

**Definition 3.2.** Let $\lfloor W \rfloor$ denote the image of $W \in \mathbf{M}[X, Y_1, \dots, Y_l]$ under

$$\begin{aligned} \mathbf{M}[X, Y_1, \dots, Y_l] \rightarrowtail \mathbf{Q}(s_1, \dots, s_l)[\![X-1]\!] &\twoheadrightarrow \mathbf{Q}(s_1, \dots, s_l), \\ f \mapsto f(X, X^{-s_1}, \dots, X^{-s_l}) &\bmod (X-1). \end{aligned}$$

**Notation.** In case $l = 1$, we just write $Y = Y_1$ and $s = s_1$.

The following generalisation of results from [7] provides the key to defining topological zeta functions via "explicit formulae" (in the sense of [6, §3]) such as those in Theorem 2.2.

**Theorem 3.3.** *Let $V_1, \dots, V_r$ be separated $\mathfrak{o}$-schemes of finite type, let $W_1, \dots, W_r \in \mathbf{M}[X, Y_1, \dots, Y_l]$, and let $S \subset \mathcal{V}_k$ be finite. If $\sum_{i=1}^{r} \# V_i(\mathfrak{O}_K/\mathfrak{P}_K) \cdot W_i(q_K, Y_1, \dots, Y_l) = 0$ for all $v \in \mathcal{V}_k \setminus S$ and all finite unramified extensions $K/k_v$, then $\sum_{i=1}^{r} \chi(V_i(\mathbf{C})) \cdot \lfloor W_i \rfloor = 0$.*

*Proof.* This follows from [28, Thm 5.12]. ♦

The topological Euler characteristics $\chi(V_i(\mathbf{C}))$ are taken with respect to an arbitrary embedding $k \hookrightarrow \mathbf{C}$. By interpreting these numbers as limits of $\# V_i(\mathfrak{O}_K/\mathfrak{P}_K)$ as "$q_K \to 1$" (see [28, §5.3]), Theorem 3.3 fits the informal description from the introduction.

## 3.2 Defining topological representation zeta functions of unipotent groups

**Lemma 3.4.** *In Theorem 2.2, we may assume that* $W_1, \dots, W_r \in \mathbf{M}[X, Y]$.

Before giving a proof of Lemma 3.4, let us use it to define topological representation zeta functions. Let $\mathbf{G}$ be a unipotent algebraic group over $k$ and let $\mathsf{G}$ be an arbitrary $\mathfrak{o}$-form of $\mathbf{G}$. By Theorem 2.2 and Lemma 3.4, there are $\mathfrak{o}$-schemes $V_1, \dots, V_r$ (separated, of finite type), rational functions $W_1, \dots, W_r \in \mathbf{M}[X, Y]$, and a finite $S \subset \mathcal{V}_k$ such that

$$\zeta_{\mathsf{G}(\mathfrak{O}_K)}(s) = \sum_{i=1}^{r} \#V_i(\mathfrak{O}_K/\mathfrak{P}_K) \cdot W_i(q_K, q_K^{-s}) \tag{3.1}$$

is an identity of analytic functions for all $v \in \mathcal{V}_k \setminus S$ and all finite extensions $K/k_v$. For all these $v$ and $K$, the rational function $\sum_{i=1}^{r} \#V_i(\mathfrak{O}_K/\mathfrak{P}_K) \cdot W_i(q_K, Y) \in \mathbf{Q}(Y)$ then only depends on $\mathsf{G}(\mathfrak{O}_K)$. Hence, by Theorem 3.3, the following definition only depends on $\mathbf{G}$ and not on the choices of $\mathsf{G}$ or the $V_1, \dots, V_r$ and $W_1, \dots, W_r$.

**Definition 3.5.** Let $\mathbf{G}$ be a unipotent algebraic group over $k$. Let $\mathsf{G}$, $V_1, \dots, V_r$, and $W_1, \dots, W_r$ be as above. The **topological representation zeta function** of $\mathbf{G}$ is

$$\zeta_{\mathbf{G},\mathrm{top}}(s) := \sum_{i=1}^{r} \chi(V_i(\mathbf{C})) \cdot \lfloor W_i \rfloor \in \mathbf{Q}(s).$$

**Example 3.6.**

(i) (See §1.) A $\mathbf{Z}$-form $\mathsf{H}$ of the Heisenberg group $\mathbf{H}$ over $\mathbf{Q}$ is given by $\mathsf{H}(R) = \left[\begin{smallmatrix} 1 & R & R \\ 0 & 1 & R \\ 0 & 0 & 1 \end{smallmatrix}\right] \leqslant \mathrm{GL}_3(R)$ for commutative rings $R$. As a special case of [34, Thm B], Voll and Stasinski proved that for all primes $p$ and all finite extensions $K/\mathbf{Q}_p$, we have $\zeta_{\mathsf{H}(\mathfrak{O}_K)}(s) = \frac{1-q_K^{-s}}{1-q_K^{1-s}} = W(q_K, q_K^{-s})$, where $W := \frac{1-Y}{1-XY} \in \mathbf{M}[X, Y]$. We conclude that $\zeta_{\mathbf{H},\mathrm{top}}(s) = \lfloor W \rfloor = s/(s-1)$.

(ii) du Sautoy [10] constructed a nilpotent Lie ring $\mathfrak{g}$ of additive rank 9 and class 2 whose ideal zeta function depends on the number $\#E(\mathbf{F}_p)$, where $E \subset \mathbf{P}^2_{\mathbf{Z}}$ is defined by $Y^2Z = X^3 - XZ^2$ (so $E \otimes_{\mathbf{Z}} \mathbf{Q}$ is an elliptic curve). Let $\mathbf{G}$ be the unipotent $\mathbf{Q}$-group corresponding to $\mathfrak{g} \otimes_{\mathbf{Z}} \mathbf{Q}$. By [37, Exc. 6], the local representation zeta functions of a particular $\mathbf{Z}$-form, $\mathsf{G}$ say, of $\mathbf{G}$ (in our terminology) are given by $\zeta_{\mathsf{G}(\mathbf{Z}_p)}(s) = W_1(p, p^{-s}) + \#E(\mathbf{F}_p) \cdot W_2(p, p^{-s})$, where $W_1 := \frac{1-Y^3}{1-X^3Y^3}$ and $W_2 := \frac{(X-1)(Y-1)Y^2}{(1-X^2Y^2)(1-X^3Y^3)}$ are elements of $\mathbf{M}[X, Y]$ and $p$ is odd. Furthermore, it is easy to see that we may replace "$\mathbf{Z}_p$" by '$\mathfrak{O}_K$" and "$p$" by "$q_K$" in this identity for any finite extension $K/\mathbf{Q}_p$. Since $\chi(E(\mathbf{C})) = 0$, we conclude that $\zeta_{\mathbf{G},\mathrm{top}}(s) = s/(s-1)$, which coincides with $\zeta_{\mathbf{H},\mathrm{top}}(s)$ from the preceding example.

While topological representation zeta functions of unipotent groups are coarser invariants than their $p$-adic counterparts, note that in our example, $\zeta_{\mathbf{G},\mathrm{top}}(s)$ nonetheless retains essential analytic properties of the $p$-adic zeta functions $\zeta_{\mathsf{G}(\mathfrak{O}_K)}(s)$.

Namely, if $s_0 \in \mathbf{C}$ is a pole ($s_0 = 1$) or zero ($s_0 = 0$) of $\zeta_{\mathbf{G},\mathrm{top}}(s)$, then $s_0$ is also a pole or zero of $\zeta_{\mathsf{G}(\mathfrak{O}_K)}(s)$, respectively. We note that for poles of topological zeta functions, a similar behaviour is a general phenomenon by [7, Thm 2.2] which provides a key motivation for studying topological zeta functions in the first place.

The preceding two examples are misleading in that both the underlying $p$-adic computations and the final results are quite simple. In general, due to the reliance (via [36, Thm 2.1]) of their proofs on the usually impractical step of constructing a principalisation of ideals, Theorem 2.2 and Lemma 3.4 do not provide us with a means of explicitly computing $p$-adic or topological representation zeta functions. Despite such theoretical obstacles, in §5, we develop a practical method for explicitly computing $\zeta_{\mathbf{G},\mathrm{top}}(s)$ at least in favourable situations. This will allow us to determine a substantial number of interesting examples of these zeta functions, see §6. Our computations of topological zeta functions include many examples with unknown associated $p$-adic zeta functions.

*Proof of Lemma 3.4.* We may assume that $\mathbf{G}$ is non-abelian. As we are free to enlarge $S$, we may further assume that $\mathsf{G}$ is an $\mathfrak{o}$-form constructed from a suitable nilpotent Lie $\mathfrak{o}$-algebra in [34, §2.1.2], see Remark 2.3. By combining [34, §§2.2–2.3], [36, §2.2], and [36, Thm 2.2], we obtain a formula

$$\zeta_{\mathsf{G}(\mathfrak{O}_K)}(s) - 1 = F(q_K) \cdot \sum_{U \subset T} c_U(q_K) \cdot (q_K - 1)^{\# U + 1} \cdot \Xi_{U,\{1\}}(q_K, \dots) \tag{3.2}$$

which is valid for almost all $v \in \mathcal{V}_k$ and all finite extensions $K/k_v$; in this formula, $F(X) = X^{-1-n(n-1)/2} \cdot \prod_{j=2}^{n-1} \frac{1-X^{-1}}{1-X^{-j}} \in \mathbf{M}[X,Y]$, where $n = \dim([\mathbf{G},\mathbf{G}])$, and the $c_U(q_K)$ are the numbers of $(\mathfrak{O}_K/\mathfrak{P}_K)$-rational points of certain quasi-projective $\mathfrak{o}$-schemes. We did not spell out the specific substitutions applied to the functions $\Xi_{U,\{1\}}$ as these are without relevance here. As indicated in the proof of [36, Prop. 2.1, pp. 1196–1197], each $\Xi_{U,\{1\}}(q_K, \dots)$ can be written as a $\mathbf{Z}$-linear combination of certain rational functions in $q_K$ and $q_K^{-s}$, each of which can be described in terms of the generating function enumerating lattice points in a rational polyhedral cone contained in $\mathbf{R}_{\geqslant 0}^U \times \mathbf{R}_{\geqslant 0}$. It follows from [29, Lem. 6.11] that if $W_U \in \mathbf{Q}(X,Y)$ is defined by $W_U(q_K, q_K^{-s}) = (q_K - 1)^{\# U+1} \cdot \Xi_{U,\{1\}}(q_K, \dots)$, then $W_U \in \mathbf{M}[X,Y]$ whence the claim follows. ♦

## 3.3 Topological representation zeta functions from perfect Lie algebras

Recall that $r_n(G)$ denotes the number of equivalence classes (in the usual sense) of continuous irreducible representations $G \to \mathrm{GL}_n(\mathbf{C})$ of a topological group $G$. It is well-known that if $G$ is compact and $p$-adic analytic, then $r_n(G) < \infty$ for all $n \geqslant 1$ if and only if $G$ is FAb, i.e. every open subgroup of $G$ has finite abelianisation. In that case, let $\zeta_G^{\mathrm{irr}}(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s}$; note that in contrast to $\zeta_G(s)$ from above, we do not take into account twisting. In [2], uniformity results for families of representation zeta functions $\zeta_G^{\mathrm{irr}}(s)$ derived from a fixed global Lie lattice were obtained. We now briefly describe how associated topological zeta functions can be obtained in the same context.

Let $\mathbf{\mathfrak{g}}$ be a finite-dimensional perfect Lie $k$-algebra and choose an arbitrary $\mathfrak{o}$-form $\mathfrak{g}$ of $\mathbf{\mathfrak{g}}$. By [2, §2.1], there exists a finite set $S \subset \mathcal{V}_k$ such that for all $v \in \mathcal{V}_k \setminus S$ and all *unramified* finite extensions $K/k_v$, we may naturally endow $\mathfrak{P}_K(\mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{O}_K)$ with the structure of a FAb $K$-analytic pro-$p_v$ group, denoted $\mathsf{G}^1(\mathfrak{O}_K)$. The analysis of the zeta functions $\zeta^{\mathrm{irr}}_{\mathsf{G}^1(\mathfrak{O}_K)}(s)$ in [2, §§3–4] and the results from [34] used above rely on the same core ingredients. In particular, the $p$-adic integrals in [2, Cor. 3.7] and [34, Cor. 2.11] are of the same shape and the key arguments from the proof of Lemma 3.4 apply to both cases. It thus follows from [2, §4] that after replacing $S$ by a finite superset, there are separated $\mathfrak{o}$-schemes $V_1, \dots, V_r$ of finite type and $W_1, \dots, W_r \in \mathbf{M}[X, Y]$ such that $\zeta_{\mathsf{G}^1(\mathfrak{O}_K)}(s) = \sum_{i=1}^r \#V_i(\mathfrak{O}_K/\mathfrak{P}_K) \cdot W_i(q_K, q_K^{-s})$ for all $v \in \mathcal{V}_k \setminus S$ and all unramified finite extensions $K/k_v$. By Theorem 3.3, we may then unambiguously define the **topological representation zeta function** of $\mathfrak{g}$ to be $\zeta^{\mathrm{irr}}_{\mathfrak{g},\mathrm{top}}(s) = \sum_{i=1}^r \chi(V_i(\mathbf{C})) \cdot \lfloor W_i \rfloor$. For example, the explicit $p$-adic formulae in [1, Thm 1.2] and [1, Thm 1.4] show that $\zeta^{\mathrm{irr}}_{\mathfrak{sl}_2(\mathbf{Q}),\mathrm{top}} = \frac{s+2}{s-1}$ and $\zeta^{\mathrm{irr}}_{\mathfrak{sl}_3(\mathbf{Q}),\mathrm{top}} = \frac{6(s+1)(s+2)}{(3s-2)(2s-1)}$.

While the focus of the present article is on the unipotent case, we note that statements analogous to the properties of topological representation zeta functions of unipotent groups derived in §4 below also hold for the functions $\zeta^{\mathrm{irr}}_{\mathfrak{g},\mathrm{top}}(s)$ (see Remark 4.9).

# 4 Fundamental properties

We derive some basic facts about topological representation zeta functions.

## Direct products

As we will now see, it suffices to consider topological representation zeta functions of unipotent groups associated with $\oplus$-indecomposable nilpotent Lie algebras.

**Lemma 4.1** (Cf. [32, §4.1.1])**.** *$\zeta_{G_1 \times G_2}(s) = \zeta_{G_1}(s) \cdot \zeta_{G_2}(s)$ for all finitely generated nilpotent pro-$p$ groups $G_1$ and $G_2$.*

*Proof.* For a topological group $H$, let $\mathrm{Irr}(H)$ denote the set of characters of its continuous, irreducible, finite-dimensional, complex representations. By reducing to the well-known case of finite groups (see e.g. [27, 8.4.2]), we obtain a bijection

$$\mathrm{Irr}(G_1) \times \mathrm{Irr}(G_2) \to \mathrm{Irr}(G_1 \times G_2), \quad (\chi_1, \chi_2) \mapsto (\chi_1 \# \chi_2 \colon (g_1, g_2) \mapsto \chi_1(g_1)\chi_2(g_2)).$$

As $(\chi_1 \# \chi_2) \cdot (\psi_1 \# \psi_2) = (\chi_1 \psi_1) \# (\chi_2 \psi_2)$ and $(\chi_1 \# \chi_2)(1) = \chi_1(1)\chi_2(1)$ for $\chi_i, \psi_i \in \mathrm{Irr}(G_i)$, we have $\tilde{r}_n(G_1 \times G_2) = \sum\limits_{d \mid n} \tilde{r}_d(G_1) \tilde{r}_{n/d}(G_2)$ whence the claim follows. ♦

**Corollary 4.2.** *$\zeta_{\mathbf{G} \times_k \mathbf{H},\mathrm{top}}(s) = \zeta_{\mathbf{G},\mathrm{top}}(s) \cdot \zeta_{\mathbf{H},\mathrm{top}}(s)$ for unipotent $k$-groups $\mathbf{G}$ and $\mathbf{H}$.* ♦

## Base extension

Passing from $p$-adic to topological representation zeta functions removes the possibly very subtle arithmetic dependence on the defining number field $k$ indicated by Theorem 2.2.

**Proposition 4.3.** *Let* $\mathbf{G}$ *be a unipotent* $k$*-group, let* $\tilde{k}/k$ *be a finite extension, and let* $\tilde{\mathbf{G}}$ *be the* $\tilde{k}$*-group obtained from* $\mathbf{G}$ *via base extension. Then* $\zeta_{\mathbf{G},\mathrm{top}}(s) = \zeta_{\tilde{\mathbf{G}},\mathrm{top}}(s)$.

*Proof.* Immediate from the stability under base extension in Theorem 2.2. ♦

We may thus unambiguously define the topological representation zeta function of a unipotent algebraic group over an algebraic closure of $\mathbf{Q}$ to be that of an arbitrary $k$-form for a suitable number field $k$.

## Restriction of scalars

Let $\mathrm{res}_{\tilde{k}/k}(-)$ denote Weil restriction from affine $\tilde{k}$-groups to affine $k$-groups.

**Proposition 4.4.** *Let* $\tilde{k}/k$ *be an extension of number fields and let* $\tilde{\mathbf{G}}$ *be a unipotent* $\tilde{k}$*-group. Then* $\zeta_{\mathrm{res}_{\tilde{k}/k}(\tilde{\mathbf{G}}),\mathrm{top}}(s) = \zeta_{\tilde{\mathbf{G}},\mathrm{top}}(s)^{|\tilde{k}:k|}$.

*Proof.* Let $l$ be a normal closure of $\tilde{k}$ over $k$ and let $\Sigma$ be the set of $k$-embeddings $\tilde{k} \to l$; note that $\#\Sigma = |\tilde{k} : k|$. For $\sigma \in \Sigma$, let $\tilde{\mathbf{G}}^\sigma$ be the $l$-group obtained from $\tilde{\mathbf{G}}$ by base change along $\sigma$. Then $\mathrm{res}_{\tilde{k}/k}(\tilde{\mathbf{G}}) \otimes_k l \approx_l \prod_{\sigma\in\Sigma} \tilde{\mathbf{G}}^\sigma$, see [22, Ch. V, §5.7] and cf. [33, 12.4.5]. The claim now follows from Corollary 4.2 and Proposition 4.3. ♦

## Behaviour at infinity

If $G$ is a finitely generated nilpotent pro-$p$ group, then $\lim\limits_{s\to+\infty} \zeta_G(s) = \tilde{r}_1(G) = 1$. This entirely trivial fact on the $p$-adic side survives passing to topological zeta functions:

**Proposition 4.5.** $\lim\limits_{s\to\infty} \zeta_{\mathbf{G},\mathrm{top}}(s) = 1$ *for every unipotent* $k$*-group* $\mathbf{G}$.

**Remark 4.6.** Unlike, say, Corollary 4.2, Proposition 4.5 is not merely a formal consequence of the analogous $p$-adic statement. As an illustration, consider

$$W(X,Y) = \frac{(X^2Y^6 + X^2Y^3 - 4XY^3 + Y^3 + 1)(1-Y)^2}{(1-X^4Y^4)(1-X^2Y^2)(1-XY)^2} \in \mathbf{M}[X,Y].$$

For $q > 1$, we have $W(q, q^{-s}) \to 1$ as $s \to +\infty$ and yet, the associated "topological zeta function" $\lfloor W \rfloor = \frac{(9s^2-6s+2)s^2}{8(s-1)^4}$ takes the value $9/8$ at infinity. Note that $W(X,Y)$ satisfies the functional equation $W(X^{-1}, Y^{-1}) = X^6 \cdot W(X,Y)$ associated with representation zeta functions of unipotent groups with 6-dimensional derived groups, see [34, Thm A].

*Proof of Proposition 4.5.* Let $\mathbf{G}$ be non-abelian. We continue to use the setting and notation from the proof of Lemma 3.4. Since $\lfloor F \rfloor = 1/(n-1)!$, we may write

$$\zeta_{\mathbf{G},\mathrm{top}}(s) - 1 = \frac{1}{(n-1)!} \sum_{U \subset T} \chi(V_U(\mathbf{C})) \cdot \sum_{a \in A_U} \gamma_U^a \lfloor W_U^a \rfloor, \tag{4.1}$$

where $V_U$ is an $\mathfrak{o}$-scheme such that $c_U(q_K) = \#V_U(\mathfrak{O}_K/\mathfrak{P}_K)$ in (3.2), each $A_U$ is finite, and for $a \in A_U$, we have $\gamma_U^a \in \mathbf{Z} \setminus \{0\}$ and $W_U^a \in \mathbf{M}[X,Y]$. The latter rational functions arise as follows. For $U \subset T$ and $a \in A_U$, we are given a full-dimensional rational simplicial cone $\mathcal{C}_U^a \subset \mathbf{R}_{\geqslant 0}^U \times \mathbf{R}_{\geqslant 0}$ (cf. the proof of [29, Lem. 6.11]) and $(q_K - 1)^{-\#U-1} W_U^a(q_K, q_K^{-s})$ is derived from the generating function enumerating lattice points within $\mathcal{C}_U^a$ by applying a certain specialisation. This specialisation is obtained by composing the one indicated by dots in (3.2) with the one implicitly given in [36, Eqn (9), p. 1193]; further details will be given below.

The remainder of the proof is devoted to showing that $\deg_s(\lfloor W_U^a \rfloor) < 0$ for all $U$ and $a$ whence the claim follows from (4.1).

The proof of [29, Lem. 6.11] shows that each $\lfloor W_U^a \rfloor$ can be written as an integer divided by a product of $\#U + 1$ factors of the form $cs + d$ for $(0, 0) \neq (c, d) \in \mathbf{Z}^2$. As in [29, Eqn (6.6)], these pairs $(c, d)$ arise in an explicit manner from the extreme rays of $\mathcal{C}_U^a$ and the given specialisation. Hence, it suffices to show that for each $U$ and $a$, $c \neq 0$ for at least one of the factors $cs + d$ associated with $\lfloor W_U^a \rfloor$. Being full-dimensional, $\mathcal{C}_U^a$ contains a vector whose last component is positive. It therefore suffices to prove the following.

($\star$) Given a 1-dimensional rational cone $\mathcal{C} \subset \mathbf{R}_{\geqslant 0}^U \times \mathbf{R}_{\geqslant 0}$ with $\mathcal{C} \cap (\mathbf{R}_{\geqslant 0}^U \times \mathbf{R}_{>0}) \neq \varnothing$, if $W \in \mathbf{Q}(X, Y)$ is obtained by applying the aforementioned specialisation to the generating function of $\mathcal{C} \cap (\mathbf{Z}^U \times \mathbf{Z})$, then $\deg_s(\lfloor (X - 1)W \rfloor) = -1$.

In order to establish ($\star$), we need to consider the relevant specialisations in some detail. We first recall a description of the $p$-adic integrals giving rise to the $\Xi_{U,\{1\}}$ in (3.2). Let $n = \dim([\mathbf{G}, \mathbf{G}])$. As in [34, §2.2.2], define matrices $\mathcal{R}(\boldsymbol{Y})$ and $\mathcal{S}(\boldsymbol{Y})$ of linear forms in $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ over $\mathfrak{o}$, and let $2u$ and $v$ denote the ranks of $\mathcal{R}(\boldsymbol{Y})$ and $\mathcal{S}(\boldsymbol{Y})$ over $k(\boldsymbol{Y})$, respectively. For $0 \leqslant i \leqslant u$, let $F_i(\boldsymbol{Y}) \subset \mathfrak{o}[\boldsymbol{Y}]$ consist of square roots of the non-zero $2i \times 2i$ principal minors of $\mathcal{R}(\boldsymbol{Y})$ (see [2, Rem. 3.6]). For $0 \leqslant j \leqslant v$, let $G_j(\boldsymbol{Y})$ be the set of non-zero $j \times j$ minors of $\mathcal{S}(\boldsymbol{Y})$. Let $K$ be a finite extension of $k_v$ for $v \in \mathcal{V}_k$. Define

$$Z_K(s) := \left(1 - q_K^{-1}\right)^2 \left(1 - q_K^{-2}\right) \cdots \left(1 - q_K^{-(n-1)}\right) (\zeta_{\mathsf{G}(\mathfrak{O}_K)}(s) - 1). \tag{4.2}$$

By combining [36, §2.2], [34, §2.2.3], and [2, §3.2], unless $v$ belongs to some finite set,

$$Z_K(s) = \int_{\mathfrak{P}_K \times \mathrm{GL}_n(\mathfrak{O}_K)} |x|_K^{s-n-1} \prod_{i=2}^{u} \frac{|x|_K^s \|F_{i-1}(\boldsymbol{y}^1)\|_K^s}{\|F_i(\boldsymbol{y}^1) \cup x F_{i-1}(\boldsymbol{y}^1)\|_K^s} \prod_{j=1}^{v} \frac{|x|_K \|G_{j-1}(\boldsymbol{y}^1)\|_K}{\|G_j(\boldsymbol{y}^1) \cup x G_{j-1}(\boldsymbol{y}^1)\|_K} \, \mathrm{d}\mu_K(x, \boldsymbol{y}), \tag{4.3}$$

where $\boldsymbol{y}^1$ denotes the first column of $\boldsymbol{y} \in \mathrm{GL}_n(\mathfrak{O}_K)$ and $\mu_K$ is the normalised Haar measure on $\mathfrak{O}_K \times \mathrm{M}_n(\mathfrak{O}_K) \approx \mathfrak{O}_K^{1+n^2}$. We observe that for $\boldsymbol{y}$ outside of the zero locus of $\prod \bigcup_{i=1}^u F_i$ in $\mathrm{GL}_n(\mathfrak{O}_K)$, the estimate $|x|_K \|F_{i-1}(\boldsymbol{y}^1)\|_K \leqslant \|F_i(\boldsymbol{y}^1) \cup x F_{i-1}(\boldsymbol{y}^1)\|_K$ shows that $Z_K(s)$ not only takes finite values for $\mathrm{Re}(s) > n$ but, in addition, $Z_K(s) \to 0$ for $\mathrm{Re}(s) \to +\infty$ as $|x|_K < 1$. By interpreting (4.3) as a specialisation of $Z_{\{1\}}$ in [36, p. 1194], the $c_U(q_K)$ and $\Xi_{U,\{1\}}$ in (3.2) are obtained by invoking [36, Thm 2.2] and, finally, the specialisation applied to the $s_\kappa$ from [36] indicated by the dots in (3.2) is defined.

To finish the proof, we consider the effect of our univariate specialisation on $\Xi_{U,\{1\}}$. Let $(m_u)_{u \in U}$ and $(n_i)_{i \in I} = (n_1)$ as in [36, Eqn (9), p. 1193]; we allow $m_u \in \mathbf{N} \cup \{0\}$ in the following so that the cone generated by $((m_u)_{u \in U}, n_1)$ is exactly of the same form as

$\mathcal{C}$ in ($\star$). Having fixed the $m_u$ and $n_1$, by applying our univariate specialisation to the exponent on the right-hand side of [36, Eqn (9)], we obtain a polynomial, $e \in \mathbf{Z}[s]$ say, of degree at most 1, corresponding to a factor $cs + d = -e$ in the discussion preceding ($\star$). By definition of the numbers $N_{u\kappa\iota}$ in [36, Eqn (8), p. 1192], the same estimate that we used to establish that $Z_K(s) \to 0$ for $\mathrm{Re}(s) \to +\infty$ shows that $e$ has degree precisely 1. This implies that ($\star$) is satisfied and completes the proof. ♦

**Corollary 4.7.** $\deg_s(\zeta_{\mathbf{G},\mathrm{top}}(s)) = 0$. ♦

In contrast, it is easy to see that the possible degrees of topological zeta functions associated with polynomials are precisely the non-positive integers.

### Location of poles

The following consequence of the proof of Proposition 4.5 is again analogous to the $p$-adic case which is proved in [32, Thm 4.24] (and also stated as [12, Thm 5.4.1]) for groups of nilpotency class 2 and $k = \mathbf{Q}$; our proof implicitly also covers the $p$-adic case without any of these two restrictions.

**Proposition 4.8.** *Let $\mathbf{G}$ be a unipotent $k$-group. If $s_0 \in \mathbf{C}$ is a pole of $\zeta_{\mathbf{G},\mathrm{top}}(s)$, then $s_0 \in \mathbf{Q}$ and $s_0 \leqslant \dim([\mathbf{G},\mathbf{G}])$.*

*Proof.* The poles of $\lfloor W \rfloor$ are rational for each $W \in \mathbf{M}[X, Y]$ by [28, Lem. 5.6(ii)] whence $s_0 \in \mathbf{Q}$. Let $s_1 \in \mathbf{Q}$ with $s_1 > n := \dim([\mathbf{G},\mathbf{G}])$. Then $Z_K(s_1)$ in (4.2) and (4.3) is finite. As $-\sum_u \nu_u m_u \leqslant 0$ in [36, Eqn (9), p. 1193], it follows that if $e \in \mathbf{Z}[s]$ is one of the polynomials in the final paragraph of the proof of Proposition 4.5 (where we now also allow $n_1 = 0$ as long as $((m_u), n_1) \neq (0, 0)$), corresponding to a factor $cs + d$ in the paragraph preceding ($\star$), then $e(s_1) < 0$. Each $\lfloor W_U^a \rfloor$ in (4.1) is therefore regular at $s_1$ whence the same is true of $\zeta_{\mathbf{G},\mathrm{top}}(s)$. ♦

**Remark 4.9.** Regarding the topological representation zeta functions $\zeta^{\mathrm{irr}}_{\mathfrak{g},\mathrm{top}}(s)$ from §3.3, the statements and proofs of Corollaries 4.2 and 4.7, and Propositions 4.3–4.5 carry over essentially verbatim. As for Proposition 4.8, if $s_0 \in \mathbf{C}$ is a pole of $\zeta^{\mathrm{irr}}_{\mathfrak{g},\mathrm{top}}(s)$, then $s_0 \in \mathbf{Q}$ and $s_0 \leqslant \dim(\mathfrak{g}) - 2$; for more refined $p$-adic bounds, see [1, Thm 1.1].

## 5 Computing topological representation zeta functions

At this point, we have encountered but one non-trivial example of a topological representation zeta function of a unipotent group, namely $\frac{s}{s-1}$ which arises from both $\mathbf{Q}$-groups in Example 3.6. In both cases, we used $p$-adic computations to derive topological formulae. We now devise a method for computing topological representation zeta functions of unipotent groups directly, i.e. without resorting to a full $p$-adic computation. As in the $p$-adic realm and the cases of previously studied types of topological zeta functions, a *general* method for computing these functions seems to invariably rely on some variation of resolution of singularities. In order to nonetheless be able to carry out explicit computations, we primarily rely on algebro-geometric genericity conditions which may or may not be satisfied in specific situations. Despite these theoretical limitations of our method, numerous previously unknown examples of topological zeta functions (see §6) demonstrate its value. Our method adapts and extends the core ingredients from [28, 29].

## 5.1 Representation data and zeta functions of unipotent groups

In this subsection, we introduce representation data. These objects provide a convenient way of encoding certain types of $p$-adic integrals, in particular those describing local representation zeta functions attached to unipotent groups as in [34, 36].

**Representation data.** We first recall some notions from convex geometry. By a **cone** in $\mathbf{R}^n$, we always mean a polyhedral one. As in [28, §3.1], a **half-open cone** in $\mathbf{R}^n$ is a set $\mathcal{C}_0 = \mathcal{C} \setminus (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r)$ for a cone $\mathcal{C} \subset \mathbf{R}^n$ and faces $\mathcal{C}_1, \dots, \mathcal{C}_r \subset \mathcal{C}$. We say that $\mathcal{C}_0$ is **rational** if $\mathcal{C}$ is rational in the usual sense or if $\mathcal{C}_0 = \varnothing$. Define the **dual** of a set $A \subset \mathbf{R}^n$ by $A^* := \{\omega \in \mathbf{R}^n : \langle \alpha, \omega \rangle \geqslant 0 \text{ for all } \alpha \in A\}$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard inner product. The dual of a finite union of rational half-open cones is a rational cone. For a cone $\mathcal{C} \subset \mathbf{R}^n$, let $k[\mathcal{C} \cap \mathbf{Z}^n]$ denote the $k$-subalgebra of $k[\boldsymbol{X}^{\pm 1}] := k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ spanned by all $\boldsymbol{X}^{\omega} := X_1^{\omega_1} \cdots X_n^{\omega_n}$ for $\omega \in \mathcal{C} \cap \mathbf{Z}^n$.

**Notation 5.1.** For $G \subset k[\boldsymbol{X}^{\pm 1}]$ and $\gamma \colon G \to \mathbf{Z}^n$, write $\boldsymbol{X}^{\gamma} G := \left\{\boldsymbol{X}^{\gamma(g)} \cdot g : g \in G\right\}$.

**Definition 5.2.** A **representation datum** over $k$ (in $n$ variables) is a quadruple

$$\mathcal{R} = (\mathcal{D}, F, \alpha, \Lambda)$$

consisting of

(i) $\mathcal{D} = \bigcup_{\delta \in \Delta} \mathcal{C}_0^{\delta}$ (finite disjoint union) for rational half-open cones $\mathcal{C}_0^{\delta} \subset \mathbf{R}^n_{\geqslant 0}$,

(ii) $F = F_1 \cup \cdots \cup F_m$ for not necessarily disjoint non-empty finite sets $F_i \subset k[\boldsymbol{X}^{\pm 1}]$,

(iii) $\alpha = (\alpha_1, \dots, \alpha_m)$ for $\alpha_i \colon F_i \to \mathbf{Z}^n$ with $\boldsymbol{X}^{\alpha_i} F_i \subset k[\mathcal{D}^* \cap \mathbf{Z}^n]$ $(i = 1, \dots, m)$, and

(iv) $\Lambda \subset \{1, \dots, m, \infty\}^2$.

By minor abuse of notation, we consider the specifications of the distinguished sets $\mathcal{C}_0^{\delta}$ and $F_i$ to be part of the definition of a representation datum. In the following, we always assume that $0 \notin F$ and $(\infty, \infty) \notin \Lambda$. We write $F_\infty = \{0\}$ and let $\alpha_\infty \colon F_\infty \to \mathbf{Z}^n$ with $\alpha_\infty(0) = 0$.

**Associated integrals.** For a non-Archimedean local field $K$, let $\nu_K$ denote the valuation on $K$ with $\nu_K(K^\times) = \mathbf{Z}$. We let $|\cdot|_K$ denote the absolute value on $K$ defined by $|x|_K = q_K^{-\nu_K(x)}$ for $x \in K$. For $\boldsymbol{x} = (x_1, \dots, x_n) \in K^n$, write $\nu_K(\boldsymbol{x}) = (\nu_K(x_1), \dots, \nu_K(x_n))$. If $A \subset K$ is finite and non-empty, we again write $\|A\|_K = \max(|a|_K : a \in A)$.

**Definition 5.3.** Let $\mathcal{R} = (\mathcal{D}, F, \alpha, \Lambda)$ be a representation datum as in Definition 5.2. Recall that $F_\infty = \{0\}$ and $\alpha_\infty = 0$. For a $p$-adic field $K \supset k$, define the "zeta function"

$$\mathsf{Z}_K^{\mathcal{R}}(\boldsymbol{s}) = \int\limits_{\mathcal{D}(K)\times\mathfrak{P}_K} \prod_{\lambda=(i,j)\in\Lambda} \|\boldsymbol{x}^{\alpha_i} F_i(\boldsymbol{x}) \cup \boldsymbol{x}^{\alpha_j} y F_j(\boldsymbol{x})\|_K^{s_\lambda} \,\mathrm{d}\mu_K(\boldsymbol{x}, y), \tag{5.1}$$

where $\boldsymbol{s} = (s_\lambda)_{\lambda\in\Lambda}$ consists of complex variables, $\mathcal{D}(K) := \{\boldsymbol{x} \in K^n : \nu_K(\boldsymbol{x}) \in \mathcal{D}\}$, and $\mu_K$ is the normalised Haar measure.

**Remark 5.4.**

(i) By Proposition 5.7 below, if $f \in k[\mathcal{D}^* \cap \mathbf{Z}^n]$ and $\boldsymbol{x} \in \mathcal{D}(K)$, then, unless $\mathfrak{o} \cap \mathfrak{P}_K$ belongs to a finite exceptional set, we have $|f(\boldsymbol{x})|_K \leqslant 1$. Hence, the assumption $\boldsymbol{X}^{\alpha_i} F_i \subset k[\mathcal{D}^* \cap \mathbf{Z}^n]$ in Definition 5.2 guarantees the finiteness of (5.1) at the very least when $\mathrm{Re}(s_\lambda) \geqslant 0$ for all $\lambda \in \Lambda$.

(ii) Note that $\mathsf{Z}_K^{\mathcal{R}}(\boldsymbol{s})$ only depends on the sets $\hat{F}_i := \boldsymbol{X}^{\alpha_i} F_i$ and not on the $F_i$ themselves. Conversely, given $\hat{F}_1, \dots, \hat{F}_m \subset k[\mathcal{D}^* \cap \mathbf{Z}^n]$, there are various ways of writing $\hat{F}_i = \boldsymbol{X}^{\alpha_i} F_i$ for $F_i$ and $\alpha_i$ as above. In Theorem 5.9, we will derive an explicit formula for a topological zeta function associated with $\mathsf{Z}_K^{\mathcal{R}}(\boldsymbol{s})$ under non-degeneracy assumptions on $F = F_1 \cup \dots \cup F_m$. The advantage of representing $\hat{F}_i$ by a pair $(F_i, \alpha_i)$ is that by carefully choosing the latter, we may be able to produce overlaps between the $F_i$ and resolve a seemingly degenerate situation.

**Connection with representation zeta functions.** Our interest in representation data is due to the following result which also explains our choice of terminology.

**Theorem 5.5** (Cf. [34, Cor. 2.11] and [36, §3.4])**.** *Let $\mathbf{G}$ be a unipotent algebraic group over $k$ and let $n = \dim([\mathbf{G}, \mathbf{G}])$. Then there exist an explicit representation datum $\mathcal{R} = (\mathcal{D}, F, \alpha, \Lambda)$ in $n$ variables as in Definition 5.2 and explicit numbers $a_\lambda, b_\lambda \in \mathbf{Z}$ with the following property: if $\mathsf{G}$ is any $\mathfrak{o}$-form of $\mathbf{G}$, there exists a finite set $S \subset \mathcal{V}_k$ such that*

$$\zeta_{\mathsf{G}(\mathfrak{O}_K)}(s) = 1 + (1 - q_K^{-1})^{-1} \mathsf{Z}_K^{\mathcal{R}}(a_\lambda s - b_\lambda)_{\lambda\in\Lambda}$$

*for all $v \in \mathcal{V}_k \setminus S$ and all finite extensions $K/k_v$. Moreover, we may assume that $\mathcal{D} = \partial\mathbf{R}^n_{\geqslant 0}$ (so that $k[\mathcal{D}^* \cap \mathbf{Z}^n] = k[\boldsymbol{X}]$) and that $F$ consists of homogeneous polynomials.*

We already made implicit use of the main ingredients of Theorem 5.5 in the proof of Proposition 4.5 even though the integrals there were of a slightly different shape, see [2, §4.1.3]. Of course, our formalism of representation data did not feature in [34, 36], but it is easy to interpret their integrals as specialisations of (5.1). Using [34, §§2.2.2–2.2.3], the representation datum $\mathcal{R}$ and family $(a_\lambda, b_\lambda)_{\lambda\in\Lambda}$ in Theorem 5.5 can be constructed from the structure constants of the Lie algebra of $\mathbf{G}$ with respect to a suitable basis.

**Associated topological zeta functions.** Let us say that a representation datum $\mathcal{R}$ is $(-1)$**-expandable** if there are separated $\mathfrak{o}$-schemes of finite type $V_1, \dots, V_r$ and rational functions $W_1, \dots, W_r \in \mathbf{M}\left[X, (Y_\lambda)_{\lambda\in\Lambda}\right]$ (see Definition 3.1) such that for almost all $v \in \mathcal{V}_k$ and all finite extensions $K/k_v$,

$$(1 - q_K^{-1})^{-1} \mathsf{Z}_K^{\mathcal{R}}(\boldsymbol{s}) = \sum_{i=1}^{r} \#V_i(\mathfrak{O}_K/\mathfrak{P}_K) \cdot W_i\Big(q_K, \big(q_K^{-s_\lambda}\big)_{\lambda\in\Lambda}\Big), \tag{5.2}$$

where $\boldsymbol{s} = (s_\lambda)_{\lambda\in\Lambda}$; the "$(-1)$" in $(-1)$-expandability reflects the exponent on the left-hand side of (5.2). If $\mathcal{R}$ is $(-1)$-expandable, then Theorem 3.3 allows us to unambiguously define the **topological zeta function** associated with $\mathcal{R}$ to be

$$\mathsf{Z}_{\mathrm{top}}^{\mathcal{R}}(\boldsymbol{s}) := \sum_{i=1}^{r} \chi(V_i(\mathbf{C})) \cdot \lfloor W_i \rfloor \in \mathbf{Q}(\boldsymbol{s}).$$

**Example 5.6.** If $\mathcal{R}$ is obtained using Theorem 5.5, then [36, Thm 2.2], [2, Eqn (4.8)], and the proof of Lemma 3.4 show that $\mathcal{R}$ is $(-1)$-expandable. Moreover, passing to topological zeta functions commutes with our univariate specialisations: by [28, Rem. 5.15], if the $(a_\lambda, b_\lambda)_{\lambda\in\Lambda}$ in Theorem 5.5 are chosen corresponding to [34, Cor. 2.11], then

$$\zeta_{\mathbf{G},\mathrm{top}}(s) - 1 = \mathsf{Z}_{\mathrm{top}}^{\mathcal{R}}(a_\lambda s - b_\lambda)_{\lambda\in\Lambda}.$$

## 5.2 Integer-valued Laurent polynomials

The techniques for computing with $p$-adic integrals developed in [28, 29] and extended here have their origin in toric geometry. The appearance of the algebra $k[\mathcal{D}^* \cap \mathbf{Z}^n]$ in Definition 5.2 is therefore perhaps not surprising since $k[\mathcal{C}^* \cap \mathbf{Z}^n]$ is the coordinate ring of the affine toric $k$-variety attached to a rational cone $\mathcal{C} \subset \mathbf{R}^n$ (see [3, Thm 1.2.18]). We now derive a simple arithmetic characterisation of $k[\mathcal{C}^* \cap \mathbf{Z}^n]$ which, in particular, implies the finiteness of the integrals in Definition 5.3 at least for $\mathrm{Re}(s_\lambda) \geqslant 0$.

Let $\mathbf{T}^n = \mathrm{Spec}(\mathbf{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$. If $R$ is a commutative ring, we identify $\mathbf{T}^n(R) = (R^\times)^n$. Let $f \in k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ and write $f = \sum_{\alpha\in\mathbf{Z}^n} c_\alpha \boldsymbol{X}^\alpha$, where $c_\alpha \in k$, almost all of which are zero. We now recall some standard terminology. The **support** of $f$ is $\mathrm{supp}(f) := \{\alpha \in \mathbf{Z}^n : c_\alpha \neq 0\}$. The convex hull of $\mathrm{supp}(f) \subset \mathbf{R}^n$ is the **Newton polytope** $\mathrm{New}(f)$ of $f$. Let $\omega \in \mathbf{R}^n$. If $f \neq 0$, then the **initial form** $\mathrm{in}_\omega(f)$ is the sum of those $c_\alpha \boldsymbol{X}^\alpha$ for $\alpha \in \mathrm{supp}(f)$ where $\langle \alpha, \omega \rangle$ is minimal; we let $\mathrm{in}_\omega(0) = 0$.

**Proposition 5.7.** *Let $k$ be a number field, $f \in k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, and let $\mathcal{C} \subset \mathbf{R}^n$ be a rational cone. Then the following are equivalent:*

*(i)* $\mathrm{New}(f) \subset \mathcal{C}^*$.

*(ii) There exists a finite set $S \subset \mathcal{V}_k$ such that for all $v \in \mathcal{V}_k \setminus S$, all finite extensions $K/k_v$, and all $\boldsymbol{x} \in \mathcal{C}(K) := \{\boldsymbol{x} \in K^n : \nu_K(\boldsymbol{x}) \in \mathcal{C}\}$, we have $\nu_K(f(\boldsymbol{x})) \geqslant 0$.*

*Proof.* Let $f \neq 0$. For $c \in K$, $\alpha \in \mathbf{Z}^n$, and $\boldsymbol{x} \in \mathbf{T}^n(K)$ with $\nu_K(\boldsymbol{x}) = \omega$, we have $\nu_K(c\boldsymbol{x}^\alpha) = \nu_K(c) + \langle \alpha, \omega \rangle$. Hence, if (i) holds and the coefficients of $f$ are $v$-adic integers for $v \in \mathcal{V}_k$, then (ii) is satisfied. Suppose that (ii) holds. Choose $v \in \mathcal{V}_k \setminus S$ such that all non-zero coefficients of $f$ are $v$-adic units and let $\omega \in \mathcal{C} \cap \mathbf{Z}^n$. Then $\mathrm{in}_\omega(f)$ has non-zero image in $\mathfrak{K}_v[\boldsymbol{X}^{\pm 1}]$. Hence, there exist a finite extension $K/k_v$ and $\boldsymbol{u} \in \mathbf{T}^n(\mathfrak{O}_K)$ with $\mathrm{in}_\omega(f)(\boldsymbol{u}) \not\equiv 0 \bmod \mathfrak{P}_K$. Choose $\pi \in \mathfrak{P}_K \setminus \mathfrak{P}_K^2$, let $\boldsymbol{x} := (\pi^{\omega_1} u_1, \dots, \pi^{\omega_n} u_n)$ and $\alpha \in \mathrm{supp}(\mathrm{in}_\omega(f))$. Then $f(\boldsymbol{x}) = \pi^{\langle \alpha, \omega \rangle}(\mathrm{in}_\omega(f)(\boldsymbol{u}) + \mathcal{O}(\pi))$ (see [28, §4.1]) and so $\nu_K(f(\boldsymbol{x})) = \langle \alpha, \omega \rangle$. By (ii), $\langle \alpha, \omega \rangle \geqslant 0$ whence $\mathrm{New}(f) \subset \{\omega\}^*$ since $\langle \alpha, \omega \rangle \leqslant \langle \beta, \omega \rangle$ for $\beta \in \mathrm{supp}(f)$. Thus, $\mathrm{New}(f) \subset \bigcap(\{\omega\}^* : \omega \in \mathcal{C} \cap \mathbf{Z}^n) =: \Delta$. Clearly, $\mathcal{C}^* \subset \Delta$. By rationality, $\mathcal{C}$ is spanned by certain non-zero $\delta_1, \dots, \delta_r \in \mathbf{Z}^n$ whence $\mathcal{C}^* = \{\delta_1\}^* \cap \dots \cap \{\delta_r\}^* \supset \Delta$. ♦

## 5.3 Reminder: computing topological subobject zeta functions

We now briefly recall a high-level description of [29, Alg. 4.1] which seeks to compute the topological zeta function associated with a "toric datum" [29, Def. 3.1] $\mathcal{T}^0$ and a substitution matrix $\beta$. Toric data are similar to the representation data considered here in that they contain convex-geometric and algebraic ingredients and that they give rise to associated $p$-adic integrals and topological zeta functions. Our method for computing topological representation zeta functions (under suitable non-degeneracy conditions) in §5.4 is built around the same core ingredients as [29, Alg. 4.1].

At the heart of [29, Alg. 4.1] lies a loop devoted to manipulating two collections of toric data: unprocessed ones which need further investigation and "regular" [29, §5.3] ones whose topological zeta functions can be computed explicitly using [28]. At first, $\mathcal{T}^0$ is the sole member of the unprocessed collection and the regular one is empty.

The aforementioned loop proceeds by successively removing a toric datum, $\mathcal{T}$ say, from the unprocessed collection. If $\mathcal{T}$ fails to be "balanced" [29, §5.2], a finite collection of derived balanced toric data is constructed and added back to the unprocessed collection, whereupon execution of the loop starts again. If $\mathcal{T}$ is balanced but not regular, then a "reduction step" [29, §7.3] attempts to construct a finite collection of toric data derived from $\mathcal{T}$ which are then added to the unprocessed collection, just as in the balancing step—it is at this point that the procedure is allowed to fail. Finally, if $\mathcal{T}$ is found to be regular, it is simply added to the regular collection. An intermediate step is concerned with "simplifications" [29, §7.2] of toric data which do not change associated topological zeta functions. The aforementioned process of removing elements from the unprocessed collection and treating them as just described is repeated until we either run out of unprocessed elements or failure is indicated by the reduction step. In the case of successful conclusion of this loop, a finite collection of regular toric data has been constructed. The topological zeta function associated with each one of these (w.r.t. the matrix $\beta$) can be explicitly computed [29, §6.7]. By taking the sum of all these rational functions, we finally recover the topological zeta function associated with $\mathcal{T}^0$ and $\beta$.

## 5.4 A method for computing topological representation zeta functions

Given a unipotent $k$-group $\mathbf{G}$, let $\mathcal{R}$ be an associated representation datum (Definition 5.2) as in Theorem 5.5. In this subsection, by drawing upon [28, 29] (see §5.3), we describe a method for explicitly computing the topological zeta function $\mathsf{Z}^{\mathcal{R}}_{\mathrm{top}}(\boldsymbol{s})$ associated with $\mathcal{R}$ under various restrictions. As explained in Example 5.6, we may then read off the topological representation zeta function $\zeta_{\mathbf{G},\mathrm{top}}(s)$ of $\mathbf{G}$.

### 5.4.1 Overview of the method

**Special case: global non-degeneracy.** We begin by sketching an important special case of our method. Given an initial representation datum $\mathcal{R} = (\mathcal{D}, F, \alpha, \Lambda)$ arising from a unipotent $k$-group $\mathbf{G}$ as above, suppose for the moment that $F$ is globally non-degenerate in the sense of [28, Def. 4.2]. That is to say, suppose that for all $G \subset F$, $\boldsymbol{x} \in \mathbf{T}^n(\bar{k})$, and $\omega \in \mathbf{R}^n$, if $\mathrm{in}_\omega(g)(\boldsymbol{x}) = 0$ (see §5.2) for all $g \in G$, then the rank of $\left[\frac{\partial \mathrm{in}_\omega(g)}{\partial X_i}(\boldsymbol{x})\right]_{g\in G, i=1,\dots,n}$ is $\#G$; here, $\bar{k}$ denotes an algebraic closure of $k$.

In order to compute $\zeta_{\mathbf{G},\mathrm{top}}(s)$, we proceed as follows. First, partition $\mathcal{D} = \bigcup_\tau \mathcal{D}_\tau$, where $\tau$ ranges over the faces of the Newton polytope $\mathcal{N}$ of $\prod F$ and $\mathcal{D}_\tau = \mathcal{D} \cap \mathrm{N}_\tau(\mathcal{N})$ (where $\mathrm{N}_\tau(\mathcal{N})$ denotes the normal cone of $\tau \subset \mathcal{N}$). Let $\mathcal{R}_\tau := (\mathcal{D}_\tau, F, \alpha, \Lambda)$. Then each $\mathsf{Z}^{\mathcal{R}_\tau}_{\mathrm{top}}(\boldsymbol{s})$ can be effectively computed via Theorem 5.9 below—in particular, thanks to global non-degeneracy of $F$, the Euler characteristics $\chi(U_G(\mathbf{C}))$ in Theorem 5.9 can be expressed in terms of convex-geometric invariants via the Bernstein-Khovanskii-Kushnirenko theorem as in [28, §§6.1–6.2]. Finally, we compute $\zeta_{\mathbf{G},\mathrm{top}}(s) = 1 + \sum_\tau \mathsf{Z}^{\mathcal{R}_\tau}_{\mathrm{top}}(a_\lambda s - b_\lambda)_{\lambda\in\Lambda}$.

In contrast to the computation of topological subobject zeta functions, the above method for cases of global non-degeneracy is already remarkably useful on its own. However, in order to compute all of the examples given in §6, the more involved approach explained in the following is required.

**General case.** In order to obtain a more powerful method which is applicable at least in some degenerate cases, we employ a more refined strategy. The high-level structure of our method coincides with [29, Alg. 4.1] as summarised in §5.3: the role of the initial toric datum $\mathcal{T}^0$ in [29] is taken by an initial representation datum (obtained via Theorem 5.5 in the cases of interest to us), and the family $(a_\lambda, b_\lambda)_{\lambda\in\Lambda}$ corresponds to the substitution matrix $\beta$ from [29]. Keeping in mind Example 5.6, it remains

(a) to define the notions of balanced and regular representation data and to provide methods for the tasks of balancing and regularity testing (§5.4.2),

(b) to describe the topological evaluation of regular representation data (§5.4.3), and

(c) to explain the simplification and reduction steps (§5.4.4).

In the following, we concisely address these points while pointing to related sections of [28, 29]. With all these ingredients in place, while the resulting method is still allowed to fail in the reduction step, in the same way that [29] extended the scope of [28], it is more powerful than the method sketched in the "special case" above which relied solely on non-degeneracy assumptions.

### 5.4.2 Balanced and regular representation data

The following is a natural substitute for the balanced and regular toric data in [29]. Henceforth, let $\bar{k}$ be a fixed algebraic closure of $k$.

**Definition 5.8** (Cf. [28, Def. 5.1, 5.5])**.** A representation datum $(\mathcal{D}, F, \alpha, \Lambda)$ with $\mathcal{D} \neq \varnothing$ is **balanced** if for each $f \in F$, the initial form $\mathrm{in}_\omega(f)$ (see §5.2) remains constant for $\omega \in \mathcal{D}$. It is **regular** if, in addition, for all $G \subset F$, if $\boldsymbol{x} \in \mathbf{T}^n(\bar{k})$ satisfies $\mathrm{in}_\omega(g)(\boldsymbol{x}) = 0$ for all $g \in G$, then the rank of $\left[\frac{\partial \mathrm{in}_\omega(g)}{\partial X_i}(\boldsymbol{x})\right]_{g\in G, i=1,\dots,n}$ is $\#G$, where $\omega \in \mathcal{D}$ is arbitrary.

The balancing procedure and regularity testing in [29, §§5.2–5.3] carry over readily.

### 5.4.3 Topological evaluation

The following theorem allows us to explicitly compute the topological zeta functions associated with regular representation data satisfying the given additional assumptions.

**Theorem 5.9.** *Let $\mathcal{R} = (\mathcal{D}, F, \alpha, \Lambda)$ be a regular representation datum in $n$ variables. Suppose that $\mathcal{D} \subset \partial\mathbf{R}^n_{\geqslant 0}$ and that each element of $F$ is homogeneous. Then there are* ***explicit*** *$k$-varieties $U_G$ and rational functions $W_G \in \mathbf{M}[X, (Y_\lambda)_{\lambda\in\Lambda}]$ indexed by subsets $G \subset F$ and a finite set $S \subset \mathcal{V}_k$ such that for all $v \in \mathcal{V}_k \setminus S$ and all finite extensions $K/k_v$,*

$$(1 - q_K^{-1})^{-1}\mathsf{Z}_K^{\mathcal{R}}(\boldsymbol{s}) = \sum_{G\subset F} \#\bar{U}_G(\mathfrak{O}_K/\mathfrak{P}_K)\cdot W_G\Big(q_K, (q_K^{-s_\lambda})_{\lambda\in\Lambda}\Big),$$

*where $\bar{\cdot}$ denotes reduction modulo $\mathfrak{p}_v$ of fixed but arbitrary $\mathfrak{o}$-models of the $U_G$. Hence, $\mathcal{R}$ is $(-1)$-expandable and $\mathsf{Z}_{\mathrm{top}}^{\mathcal{R}}(\boldsymbol{s}) = \sum\limits_{G\subset F} \chi(U_G(\mathbf{C}))\cdot\lfloor W_G\rfloor$.*

**Remark 5.10.** The assumptions on $\mathcal{D}$ and $F$ in the second sentence of Theorem 5.9 are satisfied if $\mathcal{R}$ is obtained via Theorem 5.5. Moreover, the simplification and reduction steps below are designed to preserve these properties. Finally, the validity of the univariate substitutions from Theorem 5.5 in the sense of [28, Rem. 5.15] is also preserved at all times. Hence, starting with $\mathcal{R}$ from Theorem 5.5, after successful termination of the procedure explained in §5.4.1, we have indeed computed $\zeta_{\mathbf{G},\mathrm{top}}(s) - 1 = \mathsf{Z}_{\mathrm{top}}^{\mathcal{R}}(a_\lambda s - b_\lambda)_{\lambda\in\Lambda}$.

*Proof of Theorem 5.9.* We begin by stating an analogue for representation data of the explicit $p$-adic formula in [29, Thm 5.8]. Let $\mathcal{R} = (\mathcal{D}, F, \alpha, \Lambda)$ be any regular representation datum with $\mathcal{D} \neq \varnothing$ and let $\omega \in \mathcal{D}$ be arbitrary. For $f \in F$, choose $\phi(f) \in \mathrm{supp}(\mathrm{in}_\omega(f))$. Write $\mathbf{T}_k^d := \mathbf{T}^d \otimes_{\mathbf{Z}} k$. For $G \subset F$, let $V_G$ be the subvariety of $\mathbf{T}_k^n$ defined by the vanishing of $\mathrm{in}_\omega(g)$ for $g \in G$ and the non-vanishing of $\prod_{f\in F\setminus G} \mathrm{in}_\omega(f)$. Let $(\boldsymbol{e}_g)_{g\in G}$ be the standard basis of $\mathbf{R}^G$. For $\lambda = (i,j) \in \Lambda$, define a lattice polytope

$$\mathcal{P}_{G,\lambda} = \operatorname{conv}\begin{pmatrix} (\phi(g)+\alpha_i(g), 0, \boldsymbol{e}_g), & (g \in F_i \cap G) \\ (\phi(f)+\alpha_i(f), 0, 0), & (f \in F_i \setminus G, f \neq 0) \\ (\phi(g)+\alpha_j(g), 1, \boldsymbol{e}_g), & (g \in F_j \cap G) \\ (\phi(f)+\alpha_j(f), 1, 0) & (f \in F_j \setminus G, f \neq 0) \end{pmatrix} \subset \mathcal{D}^* \times \mathbf{R}_{\geqslant 0} \times \mathbf{R}^G_{\geqslant 0}.$$

As a minor variation of [28, §3.3], given a finite union of rational half-open cones $\mathcal{H} \subset \mathbf{R}^l_{\geqslant 0}$ and a finite collection $(\mathcal{Q}_\iota)_{\iota \in I}$ of lattice polytopes $\mathcal{Q}_\iota \subset \mathcal{H}^*$, we obtain a unique $\mathscr{Z}^{\mathcal{H},(\mathcal{Q}_\iota)_{\iota\in I}} \in \mathbf{Q}(X, (Y_\iota)_{\iota \in I})$ characterised by the identity of formal power series

$$\mathscr{Z}^{\mathcal{H},(\mathcal{Q}_\iota)_{\iota\in I}} = \sum_{\omega \in \mathcal{H}\cap \mathbf{Z}^l} X^{-\langle (1,\ldots,1),\omega\rangle} \prod_{\iota \in I} Y_\iota^{\min(\langle \psi,\omega\rangle : \psi \in \mathcal{Q}_\iota)}.$$

A straight-forward modification of [28, Thm 4.10] now yields that for almost all $v \in \mathcal{V}_k$ and all finite extension $K/k_v$, we have

$$\mathsf{Z}^{\mathcal{R}}_K(\boldsymbol{s}) = \sum_{G \subset F} \#\bar{V}_G(\mathfrak{O}_K/\mathfrak{P}_K) \cdot \frac{(q_K-1)^{\#G+1}}{q_K^{n+1}} \cdot \mathscr{Z}^{\mathcal{D}\times \mathbf{R}_{>0}\times \mathbf{R}^G_{>0},(\mathcal{P}_{G,\lambda})_{\lambda\in\Lambda}}(q_K, (q_K^{-s_\lambda})_{\lambda\in\Lambda}).$$

Now assume that $\mathcal{D} \subset \partial \mathbf{R}^n_{\geqslant 0}$ and that $F$ consists of homogeneous elements. Let $\mathcal{N}$ denote the Newton polytope of $\prod F$. Since $\mathcal{R}$ is balanced, there exists a unique face $\tau \subset \mathcal{N}$ such that $\mathcal{D}$ is contained in the normal cone $\mathrm{N}_\tau(\mathcal{N})$ of $\mathcal{N}$ with respect to $\tau$, cf. [29, Lem. 5.3]. By [28, Lem. 6.1(iii)], $\tau$ coincides with the Newton polytope of $\prod_{f\in F} \mathrm{in}_\omega(f)$. Using [28, Lem. 6.1(i)], we construct varieties $U_G \subset \mathbf{T}_k^{\dim(\tau)}$ and explicit isomorphisms $V_G \approx_k U_G \times_k \mathbf{T}_k^{n-\dim(\tau)}$. Hence, the formula for $(1-q_K^{-1})^{-1}\mathsf{Z}^{\mathcal{R}}_K(\boldsymbol{s})$ in the statement of the current theorem holds for

$$W_G := X^{-n}(X-1)^{\#G+n-\dim(\tau)} \cdot \mathscr{Z}^{\mathcal{D}\times\mathbf{R}_{>0}\times\mathbf{R}^G_{>0},(\mathcal{P}_{G,\lambda})_{\lambda\in\Lambda}}(X,(Y_\lambda)_{\lambda\in\Lambda})$$

and it only remains to prove that $W_G \in \mathbf{M}[X,(Y_\lambda)_{\lambda\in\Lambda}]$. Write $\mathcal{D} = \bigcup_{\delta\in\Delta} \mathcal{C}_0^\delta$ as in Definition 5.2 for non-empty $\mathcal{C}_0^\delta$. By [29, Lem. 6.11] and since $\dim\bigl(\mathcal{C}_0^\delta \times \mathbf{R}_{>0} \times \mathbf{R}^G_{>0}\bigr) = \dim\bigl(\mathcal{C}_0^\delta\bigr) + \#G + 1$, it suffices to show that $\dim(\mathcal{C}_0^\delta) < n - \dim(\tau)$ for $\delta \in \Delta$. Fix $\delta \in \Delta$ and let $\boldsymbol{w}_1,\ldots,\boldsymbol{w}_e \in \mathcal{C}_0^\delta$ be $\mathbf{R}$-linearly independent for $e = \dim\bigl(\mathcal{C}_0^\delta\bigr)$. As $F$ consists of homogeneous elements, the vector $(1,\ldots,1)$ is contained in the closure of every normal cone of $\mathcal{N}$. In particular, there exists $\boldsymbol{w}_{e+1} \in \mathrm{N}_\tau(\mathcal{N}) \cap \mathbf{R}^n_{>0}$. By convexity, $\mathcal{C}_0^\delta \subset \partial\mathbf{R}^n_{\geqslant 0}$ is contained in a single coordinate hyperplane of $\mathbf{R}^n$. We conclude that $\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_{e+1}\} \subset \mathrm{N}_\tau(\mathcal{N})$ is linearly independent so that $\dim(\mathcal{C}_0^\delta) = e < \dim(\mathrm{N}_\tau(\mathcal{N})) = n - \dim(\tau)$. ♦

In order to explicitly compute the Euler characteristics $\chi(U_G(\mathbf{C}))$ and rational functions $\lfloor W_G \rceil$ in Theorem 5.9, we proceed similarly to [28, §3.3] and [29, §§6.5–6.7].

### 5.4.4 Simplification and reduction

**Change of generators.** Let $\mathcal{R} = (\mathcal{D}, F, \alpha, \Lambda)$ with $F = F_1 \cup \cdots \cup F_m$ as in Definition 5.2. Let $\mathcal{R}' = (\mathcal{D}', F', \alpha', \Lambda')$ with $F' = F'_1 \cup \cdots \cup F'_m$ be another representation datum in the same number of variables as $\mathcal{R}$. Suppose that $\mathcal{D} = \mathcal{D}'$, $\Lambda = \Lambda'$, and that $\mathcal{R}$ and $\mathcal{R}'$ are both $(-1)$-expandable. If $\boldsymbol{X}^{\alpha_i} F_i$ and $\boldsymbol{X}^{\alpha'_i} F'_i$ generate the same ideal of $k[\mathcal{D}^* \cap \mathbf{Z}^n]$ for $i = 1, \ldots, m$, then $\mathsf{Z}^{\mathcal{R}}_{\mathrm{top}}(\boldsymbol{s}) = \mathsf{Z}^{\mathcal{R}'}_{\mathrm{top}}(\boldsymbol{s})$. This simple observation lies at the heart of the simplification and reduction steps explained in the following. It will be most convenient to describe these operations on the level of the sets $\hat{F}_i := \boldsymbol{X}^{\alpha_i} F_i$. Specifically, beginning with $\hat{F}_1, \ldots, \hat{F}_m$, we will derive sets $\hat{F}'_1, \ldots, \hat{F}'_m$. We then obtain $\mathcal{R}'$ by constructing $F' \subset k[\boldsymbol{X}^{\pm 1}]$ of minimal cardinality, $F'_1, \ldots, F'_m \subset F'$, and $\alpha'_i \colon F'_i \to \mathbf{Z}^n$ with $\hat{F}'_i = \boldsymbol{X}^{\alpha'_i} F'_i$ for $i = 1, \ldots, m$. While $F'$ is only determined by $(\hat{F}'_1, \ldots, \hat{F}'_m)$ up to rescaling by Laurent monomials, this ambiguity not does affect questions of degeneracy, see [28, Rem. 4.3(ii)].

**Simplification.** Let $\mathcal{R} = (\mathcal{D}, F, \alpha, \Lambda)$ and $\hat{F}_i = \boldsymbol{X}^{\alpha_i} F_i$ be as above. We obtain $\hat{F}'_i$ from $\hat{F}_i$ by repeated application of the following steps. First, if $f$ divides $g$ within $k[\mathcal{D}^* \cap \mathbf{Z}^n]$ for distinct $f, g \in \hat{F}_i$, then we discard $g$ from $\hat{F}_i$. Next, if $f, f' \in \hat{F}_i$ are distinct, if $t, t'$ are terms (of initial forms, if $\mathcal{R}$ is balanced) of $f$ and $f'$, respectively, and if $t/t' \in k[\mathcal{D}^* \cap \mathbf{Z}^n]$, then we are free to replace $f$ by $g := f - f't/t'$ in $\hat{F}_i$ which we do if $|\mathrm{supp}(g)| < |\mathrm{supp}(f)|$. After finitely many iterations, this procedure stops at which point we have constructed sets $\hat{F}'_1, \ldots, \hat{F}'_m \subset k[\mathcal{D}^* \cap \mathbf{Z}^n]$. We then construct $\mathcal{R}'$ as previously explained. We note that in contrast to the simplification step for toric data in [29, §7.2], in the present setting, we cannot simply discard monomials from $F$ by modifying $\mathcal{D}$.

**Reduction.** As in [29, §7.3], the reduction step is a last resort (which might well fail) which is only ever attempted when a balanced and simplified representation datum $\mathcal{R} = (\mathcal{D}, F, \alpha, \Lambda)$ is singular (i.e. fails to be regular). Again, write $\hat{F}_i = \boldsymbol{X}^{\alpha_i} F_i$. We first construct a $\subset$-minimal set $G \subset F$ witnessing the failure of regularity in Definition 5.8. We assume that $|F_i \cap G| \geqslant 2$ for some $i$; if no such $i$ exists, then we give up right away. Define $\hat{\alpha}_i \colon F_i \to \hat{F}_i, f \mapsto \boldsymbol{X}^{\alpha_i(f)} f$ and note that $\hat{\alpha}_i$ is injective since we already applied the simplification procedure. We may therefore find distinct $f, f' \in \hat{\alpha}_i(F_i \cap G)$. We next apply an algebraic transformation to each of $f$ and $f'$ in turn, resulting in two derived representation data $\mathcal{R}^+$ and $\mathcal{R}^-$. Our ultimate goal is to remove the particular cause of singularity of $\mathcal{R}$ corresponding to $(f, f')$. Specifically, given $(f, f')$, analogously to [29, §7.3], we next consider pairs $(t, t')$ consisting of terms of $\mathrm{in}_\omega(f)$ and $\mathrm{in}_\omega(f')$, respectively, where $\omega \in \mathcal{D}$. Having chosen $(f, f')$ and $(t, t')$ (cf. [29, §7.3]), let $\gamma \in \mathbf{Z}^n$ be the exponent vector of the unique monomial in $t/t'$. Let $\mathcal{D}^+ = \mathcal{D} \cap \{\gamma\}^*$ and $\mathcal{D}^- = \mathcal{D} \setminus \mathcal{D}^+$. Define $\hat{F}^{\pm}_i$ by replacing $f$ (resp. $f'$) by $f - \frac{t}{t'} f'$ (resp. $f' - \frac{t'}{t} f$) within $\hat{F}_i$; we let $\hat{F}^{\pm}_j = \hat{F}_j$ for $j \neq i$. By proceeding as indicated above, we obtain representation data $\mathcal{R}^{\pm} = (\mathcal{D}^{\pm}, F^{\pm}, \alpha^{\pm}, \Lambda)$ and by construction, we have $\mathsf{Z}^{\mathcal{R}}_{\mathrm{top}}(\boldsymbol{s}) = \mathsf{Z}^{\mathcal{R}^+}_{\mathrm{top}}(\boldsymbol{s}) + \mathsf{Z}^{\mathcal{R}^-}_{\mathrm{top}}(\boldsymbol{s})$ (assuming $(-1)$-expandability). We then add $\mathcal{R}^+$ and $\mathcal{R}^-$ back to the unprocessed collection of our main procedure (see §5.3) and resume with the next iteration of its main loop. As in [29, §7.3], there is no guarantee that the above operations will ever succeed in removing all singularities. In order to ensure termination, we again impose a bound on the total "reduction depth".

## 6 A full classification in dimension six and further examples

We now illustrate the strength of our method for computing topological representation zeta functions of unipotent groups from §5 by presenting a large number of examples computed with its help. In particular, we give a complete list of the topological representation zeta functions associated with unipotent groups of dimension at most six over an algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. The success of our approach in these cases is in notable contrast to the computation of topological subgroup zeta functions in [28, 29] where substantial gaps remain in dimension six.

### Practical matters

In order to explicitly compute topological representation zeta functions of unipotent groups using the method explained in §5, computer assistance is indispensable for all but the smallest examples. The author's publicly available package Zeta [30] for the computer algebra system Sage [35] provides an implementation of the method from §5 for unipotent $\mathbf{Q}$-groups; all the examples discussed below were computed using it. We included references to associated $p$-adic representation zeta functions whenever we are aware of them. In all these cases, our machine computations agree with the rational functions obtained using the informal method for deducing topological zeta functions from $p$-adic ones from the introduction. Our automated approach therefore provides some independent confirmation of these manual computations. The majority of examples below are new in the sense that we are not aware of any previous computations of associated $p$-adic representation zeta functions.

### Classification by dimension

Consider the problem of determining the topological representation zeta functions of unipotent $k$-groups of dimension at most some fixed number $d$ as $k$ ranges over number fields. By invariance under base extension (Proposition 4.3) and since unipotent groups in characteristic zero are in 1-1 correspondence with finite-dimensional nilpotent Lie algebras, the topological zeta functions in question are naturally indexed by nilpotent Lie algebras of dimension at most $d$ over $\bar{\mathbf{Q}}$. By multiplicativity (Corollary 4.2), we may further restrict attention to $\oplus$-indecomposable algebras.

### A complete list in dimension at most six

A classification of 6-dimensional nilpotent Lie algebras over a field $F$ of characteristic zero was first given in [23]. The number of isomorphism classes of these algebras is finite if and only if $F^\times/(F^\times)^2$ is. In particular, the task of determining the topological representation zeta functions of unipotent $\bar{\mathbf{Q}}$-groups of dimension at most six is a finite problem. It turns out that all groups in questions are amenable to the method from §5. A complete list of the topological representation zeta functions corresponding to the 29 indecomposable nilpotent Lie algebras of dimension at most six over $\bar{\mathbf{Q}}$ is given in Table 1 (p. 26). In the column "$\mathfrak{g}$", we give the names of these algebras as in [4, §4]. For 15 of the 29 Lie algebras in Table 1, associated $p$-adic representation zeta functions have been previously computed. In cases where the author is aware of such computations, the column "$p$-adic reference" in Table 1 points to these $p$-adic results.

The column "wt" provides a measure of the "algebro-geometric complexity" of the computation of the respective topological zeta function. In detail, by the **weight** of a representation datum $(\mathcal{D}, F, \alpha, \Lambda)$, we mean the number $\sum_{f\in F}(|\mathrm{supp}(f)| - 1) \geqslant 0$. In particular, the weight of $(\mathcal{D}, F, \alpha, \Lambda)$ is zero precisely when $F$ consists entirely of Laurent monomials. The weights given in Table 1 are those of the initial representation data constructed using Theorem 5.5 (w.r.t. a suitable choice of a basis of $\mathfrak{g}$).

In the case of weight 0, the computation of topological representation zeta functions via Theorem 5.9 is purely combinatorial and immediately reduces to computing the topological incarnation of a single rational function $\mathscr{Z}^{\mathcal{H},(\mathcal{Q}_\iota)_{\iota\in I}}$ from the proof of Theorem 5.5. While this case might be devoid of algebraic geometry, it can easily be computationally expensive due to the large number of case distinctions and subdivisions of cones involved.

## Further examples

To the author's knowledge, seven is the largest dimension for which a complete classification of nilpotent Lie $\mathbf{C}$-algebras is known; see [21] for a recent comparison, including some corrections, of such classifications. Beginning in dimension seven, infinite families of pairwise non-isomorphic nilpotent Lie algebras appear over $\mathbf{C}$ and over $\bar{\mathbf{Q}}$ whence the above reduction to a finite computation no longer applies. In addition, there are 7-dimensional nilpotent Lie $\bar{\mathbf{Q}}$-algebras which are not amenable to our method. Despite these limitations, we can compute a large number of interesting examples of topological representation zeta functions of unipotent $\bar{\mathbf{Q}}$-groups of dimension seven and beyond, and Table 2 (p. 27) includes a selection of these. To the author's knowledge, no $p$-adic representation zeta functions are known for any of the examples in Table 2. The first batch of Lie algebras in Table 2 is taken from [31]. The algebras $N_i^{8,d}$ are taken from the lists of 8-dimensional Lie $\mathbf{C}$-algebras of class 2 with $d$-dimensional centre in [26, 39]. The remaining algebras in Table 2 are obtained from algebras in Table 1 by base change to dual numbers. In detail, let $k[\varepsilon] = k[X]/X^2$ and for a $k$-algebra $\mathfrak{g}$, let $\mathfrak{g}[\varepsilon] = \mathfrak{g} \otimes_k k[\varepsilon]$ regarded as a $k$-algebra. For example, $L_{3,2}$ is the Heisenberg Lie algebra and $L_{3,2}[\varepsilon] \approx L_{6,22}(0)$. Recall from Propositions 4.3 and 4.4 that base extension of number fields followed by restriction of scalars is very well-behaved on the level of topological representation zeta functions. The examples in Table 2 show that this is not generally true for the operation $\mathfrak{g} \mapsto \mathfrak{g}[\varepsilon]$ (but see Question 7.3 below). For yet more examples, we refer the reader to the database of topological representation zeta functions included with [30].

## 7 Open questions

Based on our experimental evidence, we state some open questions which might provide interesting avenues for future research. Throughout, let $\mathbf{G}$ be a non-abelian unipotent $k$-group. All examples of topological representation zeta functions that we computed are consistent with Questions 7.1–7.5 below having positive answers; using the explicit $p$-adic formulae from [34, Thm B], this includes examples in much higher dimensions than those covered by Tables 1–2.

By Corollary 4.7, $\zeta_{\mathbf{G},\mathrm{top}}(s)$ has degree zero in $s$. In contrast, no explanation of the observed degrees seems to be known for topological subalgebra zeta functions, see [28, §8]; note that in the enumeration of subalgebras, passing from $p$-adic to topological zeta functions involves an additional transformation [28, Ex. 5.11(iii), Def. 5.17]. Perhaps the following question is a more appropriate analogue of [28, Conj. I].

**Question 7.1.** Does $\zeta_{\mathbf{G},\mathrm{top}}(s) - 1$ always have degree $-1$ in $s$?

We now consider a refinement of Question 7.1 in the spirit of [29, §9.3]. Define

$$\omega(\mathbf{G}) := s(\zeta_{\mathbf{G},\mathrm{top}}(s) - 1)\Big|_{s=\infty} = s^{-1}\big(\zeta_{\mathbf{G},\mathrm{top}}(s^{-1}) - 1\big)\Big|_{s=0} \in \mathbf{Q}.$$

Question 7.1 has a positive answer if and only if always $\omega(\mathbf{G}) \neq 0$. For example, the groups with Lie algebras $1,4,5,7_B$, $1,3,7_B$, and $1,2,3,5,7_C$ in Table 2 have $\omega$-invariant $\frac{7}{3}$.

**Question 7.2.** Is $\omega(\mathbf{G})$ always positive?

It would be interesting to find a group-theoretic interpretation of $\omega(\mathbf{G})$. Using Corollary 4.2 and Proposition 4.5, it is easy to see that if $\mathbf{H}$ is another unipotent $k$-group, then $\omega(\mathbf{G} \times_k \mathbf{H}) = \omega(\mathbf{G}) + \omega(\mathbf{H})$. Let $\mathbf{G}[\varepsilon]$ denote the $k$-group attached to $\mathfrak{g}[\varepsilon] = \mathfrak{g} \otimes_k k[\varepsilon]$ from §6, where $\mathfrak{g}$ is the Lie algebra of $\mathbf{G}$ and $k[\varepsilon] = k[X]/X^2$.

**Question 7.3.** Do we always have $\omega(\mathbf{G}[\varepsilon]) = \frac{3}{2}\omega(\mathbf{G})$?

For example, the groups with Lie algebras $L_{6,24}(0)$ and $L_{6,24}(0)[\varepsilon]$ in Tables 1–2 have $\omega$-invariants $\frac{5}{2}$ and $\frac{15}{4}$, respectively. Looking at the various examples in Tables 1–2, one cannot help but notice factors $s^e$ ($e \geqslant 1$) in the numerators of all functions given there.

**Question 7.4.** Does $\zeta_{\mathbf{G},\mathrm{top}}(s)$ always vanish at zero?

More generally, we may ask for an interpretation of the precise order of vanishing of $\zeta_{\mathbf{G},\mathrm{top}}(s)$ at zero.

Finally, fixed points of topological representation zeta functions seem to exhibit properties similar to zeros of topological subalgebra zeta functions, cf. [28, Conj. III].

**Question 7.5.** Let $s_0 \in \mathbf{C}$ with $\zeta_{\mathbf{G},\mathrm{top}}(s_0) = s_0$. Do we have $0 \leqslant \mathrm{Re}(s_0) \leqslant \dim(\mathbf{G}) - 1$?

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| $\mathfrak{g}$ | dim | $\zeta_{\mathbf{G},\mathrm{top}}(s)$ | wt | $p$-adic reference |
|---:|:---:|:---|:---:|---:|
| abelian | 1 | $1$ | 0 | obvious |
| $L_{3,2}$ | 3 | $\frac{s}{s-1}$ | 0 | [25, Thm 5] |
| $L_{4,3}$ | 4 | $\frac{s^2}{(s-1)^2}$ | 0 | $M_3$ [12, (4.2.24)] |
| $L_{5,4}$ | 5 | $\frac{2s}{2s-1}$ | 0 | $B_4$ [32, Ex. 6.3] |
| $L_{5,5}$ | 5 | $\frac{(2s-1)s}{2(s-1)^2}$ | 0 | $G_{5,3}$ [12, Tab. 5.2] |
| $L_{5,6}$ | 5 | $\frac{2s^2}{(2s-3)(s-1)}$ | 0 | |
| $L_{5,7}$ | 5 | $\frac{s^2}{(s-1)(s-2)}$ | 0 | $M_4$ [12, (4.2.24)] |
| $L_{5,8}$ | 5 | $\frac{s}{s-2}$ | 0 | $M_{3,3}$ [12, (5.3.7)]; $G_3$ [32, Ex. 6.2] |
| $L_{5,9}$ | 5 | $\frac{s^2}{(s-1)(s-2)}$ | 0 | $F_{3,2}$ [12, Tab. 5.2] |
| $L_{6,10}$ | 6 | $\frac{2s^2}{(2s-1)(s-1)}$ | 0 | $G_{6,12}$ [12, Tab. 5.2] |
| $L_{6,11}$ | 6 | $\frac{(6s+1)s}{2(3s-4)(s-1)}$ | 0 | |
| $L_{6,12}$ | 6 | $\frac{2s^2}{(2s-3)(s-1)}$ | 0 | |
| $L_{6,13}$ | 6 | $\frac{(12s^2-18s+7)s^2}{6(2s-1)(s-1)^3}$ | 1 | |
| $L_{6,14}$ | 6 | $\frac{(12s^2-12s+1)s^2}{3(2s-1)(2s-3)(s-1)^2}$ | 1 | |
| $L_{6,15}$ | 6 | $\frac{(6s-7)s^2}{(3s-5)(2s-3)(s-1)}$ | 1 | |
| $L_{6,16}$ | 6 | $\frac{2s^3}{(2s-1)(s-1)(s-2)}$ | 0 | |
| $L_{6,17}$ | 6 | $\frac{(2s-3)s^2}{2(s-1)(s-2)^2}$ | 0 | |
| $L_{6,18}$ | 6 | $\frac{s^2}{(s-1)(s-3)}$ | 0 | $M_5$ [12, (4.2.24)] |
| $L_{6,19}(0)$ | 6 | $\frac{s^2}{(s-1)(s-2)}$ | 0 | $G_{6,7}$ [12, Tab. 5.2] |
| $L_{6,19}(1)$ | 6 | $\frac{2s^2}{(2s-1)(s-2)}$ | 0 | $G_{6,14}$ [12, Tab. 5.2] |
| $L_{6,20}$ | 6 | $\frac{(2s-1)s}{2(s-1)(s-2)}$ | 0 | |
| $L_{6,21}(0)$ | 6 | $\frac{s^2}{(s-2)^2}$ | 0 | |
| $L_{6,21}(1)$ | 6 | $\frac{2s^2}{(2s-3)(s-2)}$ | 0 | |
| $L_{6,22}(0)$ | 6 | $\frac{2s}{2s-3}$ | 0 | [32, Ex. 6.5] |
| $L_{6,23}$ | 6 | $\frac{(2s-3)s}{2(s-2)^2}$ | 0 | |
| $L_{6,24}(0)$ | 6 | $\frac{(4s^2-6s+1)s}{(2s-3)^2(s-1)}$ | 0 | |
| $L_{6,24}(1)$ | 6 | $\frac{(2s+1)s}{(2s-3)(s-1)}$ | 1 | |
| $L_{6,25}$ | 6 | $\frac{(s-1)s}{(s-2)^2}$ | 0 | $M_{4,3}$ [12, (5.3.7)] |
| $L_{6,26}$ | 6 | $\frac{s}{s-3}$ | 0 | $F_{1,1}$ [34, Thm B] |

Table 1: Topological representation zeta functions associated with $\oplus$-indecomposable nilpotent Lie algebras of dimension at most six (complete list)

| $\mathfrak{g}$ | dim | $\zeta_{\mathbf{G},\mathrm{top}}(s)$ | wt |
|---:|:---:|:---|:---:|
| $3,5,7_C$ [31, p. 483] | 7 | $\frac{(2s-3)(s-1)s}{2(s-2)^3}$ | 1 |
| $2,7_B$ [31, p. 484] | 7 | $\frac{s}{s-1}$ | 0 |
| $2,5,7_D$ [31, p. 484] | 7 | $\frac{4(s-1)s}{(2s-3)^2}$ | 0 |
| $2,5,7_G$ [31, p. 484] | 7 | $\frac{(12s^2-22s+9)s}{2(3s-4)(2s-3)(s-1)}$ | 1 |
| $2,4,7_J$ [31, p. 485] | 7 | $\frac{(6s+1)s}{2(3s-4)(s-2)}$ | 1 |
| $2,4,7_R$ [31, p. 486] | 7 | $\frac{(6s^2-14s+7)s}{2(3s-4)(s-2)^2}$ | 2 |
| $2,4,5,7_K$ [31, p. 487] | 7 | $\frac{2(24s^3-82s^2+84s-23)s}{(4s-7)(3s-5)(2s-3)^2}$ | 1 |
| $2,3,5,7_A$ [31, p. 488] | 7 | $\frac{2s^2}{(2s-3)(s-2)}$ | 3 |
| $2,3,5,7_D$ [31, p. 488] | 7 | $\frac{(36s^2-24s-29)s}{2(3s-4)^2(2s-3)}$ | 2 |
| $2,3,4,5,7_D$ [31, p. 489] | 7 | $\frac{(6s^3-22s^2+25s-8)s^2}{2(3s-5)(s-1)^2(s-2)^2}$ | 2 |
| $2,3,4,5,7_E$ [31, p. 489] | 7 | $\frac{2s^2}{(2s-5)(s-2)}$ | 3 |
| $1,5,7$ [31, p. 489] | 7 | $\frac{(3s-1)s}{(3s-2)(s-1)}$ | 0 |
| $1,4,7_C$ [31, p. 489] | 7 | $\frac{3(s-1)s}{(3s-4)(s-3)}$ | 0 |
| $1,4,5,7_B$ [31, p. 490] | 7 | $\frac{(3s-2)s^2}{3(s-1)^3}$ | 0 |
| $1,3,7_B$ [31, p. 490] | 7 | $\frac{(9s^3-9s^2+3s-1)s}{(3s-2)^2(s-1)^2}$ | 0 |
| $1,3,5,7_M:0$ [31, p. 491] | 7 | $\frac{(12s^2-42s+37)(s-1)s}{6(2s-3)(s-2)^3}$ | 1 |
| $1,3,4,5,7_I$ [31, p. 492] | 7 | $\frac{(3s-4)s^2}{(3s-5)(s-2)^2}$ | 1 |
| $1,2,4,5,7_E$ [31, p. 493] | 7 | $\frac{(288s^3-624s^2+266s+81)s}{(6s-7)(4s-5)^2(3s-4)}$ | 2 |
| $1,2,4,5,7_J$ [31, p. 493] | 7 | $\frac{2(6s-7)s^2}{3(2s-3)^2(s-2)}$ | 2 |
| $1,2,3,5,7_B$ [31, p. 494] | 7 | $\frac{(8s^2-12s+5)s^2}{8(s-1)^4}$ | 3 |
| $1,2,3,5,7_C$ [31, p. 494] | 7 | $\frac{(3024s^5-10248s^4+13286s^3-7893s^2+1900s-64)s}{56(9s-8)(6s-5)(s-1)^4}$ | 3 |
| $1,2,3,4,5,7_F$ [31, p. 494] | 7 | $\frac{(3s-4)(2s-3)s^2}{2(3s-5)(s-1)(s-2)^2}$ | 3 |
| $1,2,3,4,5,7_G$ [31, p. 494] | 7 | $\frac{(4s-5)s^2}{(4s-7)(s-1)(s-2)}$ | 2 |
| $N_1^{8,2}$ [26, Thm 1] | 8 | $\frac{6s^2}{(3s-2)(2s-1)}$ | 1 |
| $N_5^{8,2}$ [26, Thm 1] | 8 | $\frac{6s}{6s-5}$ | 0 |
| $N_3^{8,3}$ [39, Thm 3.2] | 8 | $\frac{(4s^2-6s+1)s}{(2s-3)^2(s-1)}$ | 0 |
| $N_7^{8,3}$ [39, Thm 3.2] | 8 | $\frac{(6s-7)s}{(3s-5)(2s-3)}$ | 1 |
| $N_1^{8,4}$ [39, Thm 3.3] | 8 | $\frac{(s-1)s}{(s-2)(s-3)}$ | 1 |
| $N_3^{8,4}$ [39, Thm 3.3] | 8 | $\frac{2(s-2)s}{(2s-5)(s-3)}$ | 1 |
| $L_{4,3}[\varepsilon]$ | 8 | $\frac{2(4s^2-6s+1)s}{(2s-3)^3}$ | 1 |
| $L_{5,4}[\varepsilon]$ | 10 | $\frac{4s}{4s-3}$ | 0 |
| $L_{5,5}[\varepsilon]$ | 10 | $\frac{16(s-1)^2s}{(4s-5)(2s-3)^2}$ | 2 |
| $L_{5,7}[\varepsilon]$, $L_{5,9}[\varepsilon]$ | 10 | $\frac{(2s^2-4s+1)s}{(2s-5)(s-2)^2}$ | 3 |
| $L_{5,8}[\varepsilon]$ | 10 | $\frac{(2s-3)s}{(2s-5)(s-2)}$ | 1 |
| $L_{6,22}(0)[\varepsilon] \approx L_{3,2}[\varepsilon][\varepsilon']$ | 12 | $\frac{2(2s-3)s}{(4s-7)(s-2)}$ | 2 |
| $L_{6,23}[\varepsilon]$ | 12 | $\frac{2(8s^3-44s^2+82s-53)s}{(4s-9)(2s-5)^2(s-2)}$ | 5 |
| $L_{6,24}(0)[\varepsilon]$ | 12 | $\frac{2(512s^6-4544s^5+16544s^4-31500s^3+32885s^2-17685s+3769)s}{(8s-15)(4s-7)^3(2s-3)(s-2)^2}$ | 5 |
| $L_{6,25}[\varepsilon]$ | 12 | $\frac{2(4s^3-20s^2+33s-19)s}{(2s-5)^3(s-2)}$ | 3 |
| $L_{6,26}[\varepsilon]$ | 12 | $\frac{2(s-2)s}{(2s-7)(s-3)}$ | 3 |

Table 2: Examples of topological representation zeta functions in dimension $\geqslant 7$