Computing topological zeta functions of groups, algebras, and modules, II

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Building on our previous work, we develop the first practical algorithm for computing topological zeta functions of nilpotent groups, possibly non-associative algebras, and modules. While we previously depended upon non-degeneracy assumptions, the theory developed here allows us to overcome these restrictions in various interesting cases.

1 Introduction

Topological zeta functions. A recent addition to the theory of zeta functions of algebraic structures, topological zeta functions of groups and algebras were introduced by du Sautoy and Loeser [14] as asymptotic invariants related to the enumeration of subobjects. They are limits as the "prime tends to one" of the local subobject zeta functions due to Grunewald, Segal, and Smith [17] in the same way that topological zeta functions of polynomials due to Denef and Loeser [11] are limits of Igusa's local zeta functions [19]. Topological zeta functions of polynomials have been investigated extensively (see, for instance, [24, 33, 34]). A key motivation for their study is given by connections with Igusa's monodromy conjecture; see [11, §3.3] and [9, §4.3].

A decade after their introduction, apart from a short list of examples in [14, §9], topological zeta functions of groups, algebras, and modules remained uncharted territory. It is the purpose of the project begun in [27] and continued here to change that. In contrast to the realm of Igusa's local zeta function, analytic properties of (local or global) subobject zeta functions are generally poorly understood. In [27], the author stated four conjectures on analytic properties of topological zeta functions arising from the

enumeration of subgroups, subalgebras, and submodules; for another related conjecture, see Conjecture V below. Thanks to [27] and the methods developed in the present article, there exists extensive experimental evidence in support of these conjectures. Apart from promoting the study of topological subobject zeta functions as a subject of independent interest, if confirmed, some of the author's conjectures would have implications for local subobject zeta functions (see [27, §8]).

We now give an informal explanation of the limit $p \to 1$ used to define topological subobject zeta functions. Let G be a finitely generated torsion-free nilpotent group. Let $a_n(G)$ denote the number of subgroups of index n of G. Recall that the local subgroup zeta function $\zeta_{G,p}(s)$ of G at the prime p is given by the Dirichlet series $\zeta_{G,p}(s) := \sum_{e=0}^{\infty} a_{p^e}(G)p^{-es}$. Let d be the Hirsch length of G. Informally, the topological subgroup zeta function $\zeta_{G,\mathrm{top}}(s)$ of G is the rational function in s obtained as the constant term of $(1 - p^{-1})^d \zeta_{G,p}(s)$ as a series in $p - 1$; here, the expansion as a series in $p - 1$ is obtained using the binomial series for $p^{-s} = (1 + (p-1))^{-s}$. For example, it is well-known that $\zeta_{\mathbf{Z}^d,p}(s) = \frac{1}{(1-p^{-s})(1-p^{1-s})\cdots(1-p^{d-1-s})}$ and we find that $\zeta_{\mathbf{Z}^d,\mathrm{top}}(s) = \frac{1}{s(s-1)\cdots(s-(d-1))}$.

Central objects: toric data. At the heart of the present article lies the notion of a *toric datum*. A toric datum consists of a half-open cone within some Euclidean space and a finite collection of Laurent polynomials. We will begin our study of toric data in §3, where we will also relate them to the cone integral data of du Sautoy and Grunewald [13]. As we will see, toric data give rise to associated p-adic integrals (closely related to the cone integrals from [13]) and to topological zeta functions by means of a limit $p \to 1$; this limit extends the process of taking the constant term of a series in $p - 1$ from the informal description given above. Most importantly, topological zeta functions arising from the enumeration of subgroups, subalgebras, and submodules can be expressed in terms of toric data.

In general, the computation of such zeta functions relies on the same impractical ingredient as the computation of cone integrals: resolution of singularities. In suitably non-degenerate settings, explicit resolutions can be obtained via so-called toric modifications [25, Ch. II]. Based on such classical results from toric geometry and previous applications to Igusa-type zeta functions ([10], in particular), [27] provides us with explicit convex-geometric formulae for topological zeta functions associated with toric data under non-degeneracy assumptions. Unfortunately, when it comes to the computation of topological subgroup, subalgebra, or submodule zeta functions, the practical scope of [27] on its own is limited: more often than not, the non-degeneracy assumptions are violated for examples of interest.

Main result. Our main result, Algorithm 4.1, is a practical algorithm which seeks to compute topological zeta functions associated with toric data in favourable situations, significantly extending the applicable range of the ideas in [27]. The practicality of Algorithm 4.1 is demonstrated by a computer implementation [28] which also includes a database containing a substantial number of topological subalgebra and ideal zeta functions computed using Algorithm 4.1. These computations provide strong evidence for

the intriguing features of such topological zeta functions predicted by the conjectures in [27, §8]. Algorithm 4.1 is based on a series of algebraic and convex-geometric operations, a subset of which constitutes an algorithmic version of [27, Thm 6.7]. By adding further steps, we extend the scope of Algorithm 4.1 beyond the non-degeneracy assumptions of [27]. As an illustration, the computation of the topological subalgebra zeta function of the nilpotent Lie ring Fil_4 in §9.1 makes essential use of the new ingredients in this article and thus demonstrates the strength of Algorithm 4.1.

Some of the steps employed in Algorithm 4.1 (such as *balancing* in §5.2) can be interpreted within the geometric framework of toric modifications. Others (e.g. *simplification* in §7.2) are most naturally regarded as systematic and generalised versions of "tricks" for the evaluation of p-adic integrals previously employed by Woodward [36] in a semi-automatic fashion. Yet others (such as *reduction* in §7.3) are inspired by the theory of Gröbner bases.

Previous computations. While Algorithm 4.1 is the first of its kind specifically designed to compute the *topological* zeta functions considered here, a significant number of *local* zeta functions of groups and algebras have been previously computed; see e.g. [15, 37]. Although the informal definition of topological zeta functions from above suggests that they can be deduced from local formulae, a rigorous approach requires additional information; cf. [14, §§7,9.3] and see [27, Rem. 5.20].

A significant proportion of the known local zeta functions of groups and algebras were found by Woodward using a combination of machine-computations and human insight [36]. The number of zeta functions he managed to compute is particularly impressive in view of the relatively elementary nature of his method which is based on skillful (but ultimately ad hoc) applications of certain tricks [36, §2.3.2] for computing with p-adic integrals. Unfortunately, due to the reliance of his computations on human guidance, they are difficult to reproduce.

Although, as we mentioned before, some of Woodward's tricks can be regarded as special cases of the simplification step in §7.2, the p-adic part of our method does not generalise his approach. In particular, using Algorithm 4.1, we managed to determine topological zeta functions whose local versions Woodward could not compute (see §9.1) and, conversely, there are examples of local zeta functions computed by him whose topological counterparts cannot be determined using Algorithm 4.1.

The techniques developed in the present article are specifically designed for the computation of topological zeta functions of groups, algebras, and modules. However, through our use of [27], we draw heavily upon formulae for Igusa-type zeta functions and associated topological zeta functions that have been obtained under non-degeneracy assumptions; see, in particular, [10, 11, 35].

Outline. After a brief reminder on local zeta functions of groups, algebras, and modules in §2, we introduce toric data and associated p-adic integrals in §3. The central section of the present article is §4 which is devoted to describing our main algorithm and its various components; details on the latter will be provided in subsequent sections. In §5, we

consider toric data which are balanced and regular—these two notions provide the main link between the present article and its predecessor [27]. In particular, as we will explain in §6, topological zeta functions associated with regular toric data can be computed using [27]. In order to keep the present article reasonably self-contained, key facts from [27] will be briefly recalled as needed. The two remaining ingredients of Algorithm 4.1, namely simplification and reduction are discussed in §7. Practical aspects and the author's implementation Zeta [28] of Algorithm 4.1 are briefly discussed in §8. Finally, in §9, we consider specific examples which illustrate key steps of Algorithm 4.1 and which also demonstrate its practical strength.

Acknowledgements

Notation

The following notation is used repeatedly:

k	a number field
$\mathfrak{o}$	the ring of integers of k
$\subset$	not necessarily proper inclusion
$\sqcup$	concatenation of lists
$\mathbf{N}, \mathbf{N}_0$	$\{1, 2, \dots\}$, $\{0, 1, \dots\}$
$\mathbf{Z}, \mathbf{R}, \mathbf{C}$,	the integers, real numbers, and complex numbers
$\mathbf{Z}_p, \mathbf{Q}_p$	the p-adic integers and p-adic numbers
K	a p-adic field (that is, a finite extension of $\mathbf{Q}_p$) endowed with an embedding $k \subset K$
$\mathfrak{O}_K, \mathfrak{P}_K, \mathfrak{p}_K$	the valuation ring of K, maximal ideal of $\mathfrak{O}_K$, and $\mathfrak{o} \cap \mathfrak{P}_K$
q_K	$\#(\mathfrak{O}_K/\mathfrak{P}_K)$
π_K	a fixed uniformiser of K
$\bar{V}$	the reduction modulo $\mathfrak{p}_K$ of a k-variety V (see [27, §5.2])
ν_K	the valuation on K with $\nu_K(\pi_K) = 1$; we write $\nu(\boldsymbol{x}) := \big(\nu_K(x_1), \dots, \nu_K(x_n)\big)$ for $\boldsymbol{x} = (x_1, \dots, x_n) \in K^n$
$\lvert x \rvert_K$	$q_K^{-\nu_K(x)}$ for $x \in K$
$\lVert M \rVert_K$	$\sup\big(\lvert x \rvert_K : x \in M\big)$ for $M \subset K$
μ_K	the Haar measure on K^n with $\mu_K(\mathfrak{O}_K^n) = 1$
$R[\![z]\!]$	the ring of formal power series in z over R
$F(\!(z)\!)$	the field of fractions of $F[\![z]\!]$, where F is a field
$R^\times$	the unit group of the ring R
$\mathbf{T}^n, \mathbf{T}^n_R$	$\mathrm{Spec}\big(\mathbf{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]\big)$, $\mathbf{T}^n \times \mathrm{Spec}(R)$; we identify $\mathbf{T}^n(R) = (R^\times)^n$
$\boldsymbol{X}, \boldsymbol{X}^\alpha$	$(X_1, \dots, X_n)$, $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$

2 Background: zeta functions of groups, algebras, and modules

The following is an abridged version of [27, §2]. We investigate the following mild generalisations of subring, ideal, and submodule zeta functions; cf. [17, 29].

Definition 2.1. Let R be the ring of integers in a number field or in a p-adic field.

  (i) Let M be a free R-module of finite rank and let $\mathcal{E}$ be a unital R-subalgebra of $\mathrm{End}_R(M)$. The **submodule zeta function** of $\mathcal{E}$ acting on M is

$$\zeta_{\mathcal{E} \curvearrowright M}(s) := \sum_{n=1}^{\infty} \#\{U : U \text{ is an } \mathcal{E}\text{-submodule of } M \text{ with } |M : U| = n\} \cdot n^{-s}.$$

  (ii) Let $\mathcal{A}$ be a possibly non-associative R-algebra whose underlying R-module is free and of finite rank. The **subalgebra zeta function** of $\mathcal{A}$ is

$$\zeta_{\mathcal{A}}(s) := \sum_{n=1}^{\infty} \#\{\mathcal{U} : \mathcal{U} \text{ is an } R\text{-subalgebra of } \mathcal{A} \text{ with } |\mathcal{A} : \mathcal{U}| = n\} \cdot n^{-s}.$$

Let $\Omega(\mathcal{A})$ be the unital R-subalgebra of $\mathrm{End}_R(\mathcal{A})$ generated by all maps $x \mapsto ax$ and $x \mapsto xa$ for $a \in \mathcal{A}$. The **ideal zeta function** of $\mathcal{A}$ is $\zeta_{\mathcal{A}}^{\triangleleft}(s) := \zeta_{\Omega(\mathcal{A}) \curvearrowright \mathcal{A}}(s)$.

We refer to (possibly non-associative) $\mathbf{Z}$-algebras as (possibly non-associative) rings, to subalgebras as subrings, etc. Let $\mathrm{Tr}_d(R)$ be the ring of upper triangular $d \times d$-matrices over R. The following result of du Sautoy and Grunewald is stated as in [27]. For the remainder of this article, k is a number field with ring of integers $\mathfrak{o}$.

Theorem 2.2 ([13, §5])**.**

  (i) *Let $\mathcal{A}$ be a possibly non-associative $\mathfrak{o}$-algebra which, as an $\mathfrak{o}$-module, is free and of rank d. For a p-adic field $K \supset k$, we identify $\mathrm{Tr}_d(K) \approx K^{\binom{d+1}{2}}$ and we regard $\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O}_K$ as an $\mathfrak{O}_K$-algebra. There exists a finite set $\boldsymbol{f} \subset \mathfrak{o}[X_{ij}^{\pm 1} : 1 \leqslant i \leqslant j \leqslant d]$ with the following property: if $K \supset k$ is a p-adic field, then*

$$\zeta_{\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O}_K}(s) = (1 - q_K^{-1})^{-d} \int_{\left\{\boldsymbol{x} \in \mathrm{Tr}_d(\mathfrak{O}_K) : \|\boldsymbol{f}(\boldsymbol{x})\|_K \leqslant 1\right\}} \prod_{i=1}^{d} |x_{ii}|_K^{s-i} \, \mathrm{d}\mu_K(\boldsymbol{x}). \qquad (2.1)$$

  (ii) *Let M be a free $\mathfrak{o}$-module of rank d. Let $\mathcal{E}$ be a unital $\mathfrak{o}$-subalgebra of $\mathrm{End}_{\mathfrak{o}}(M)$. There are Laurent polynomials as in (i) such that the conclusion of (i) holds for $\zeta_{(\mathcal{E} \otimes_{\mathfrak{o}} \mathfrak{O}_K) \curvearrowright (M \otimes_{\mathfrak{o}} \mathfrak{O}_K)}(s)$ in place of $\zeta_{\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O}_K}(s)$.*

As for subgroups, the enumeration of normal subgroups of a torsion-free finitely generated nilpotent group G gives rise to local normal subgroup zeta functions $\zeta_{G,p}^{\triangleleft}(s)$.

Theorem 2.3 ([17, §4])**.** *Let G be a finitely generated torsion-free nilpotent group with associated Lie $\mathbf{Q}$-algebra $\mathfrak{L}(G)$ under the Mal'cev correspondence. Let $\mathcal{L} \subset \mathfrak{L}(G)$ be a $\mathbf{Z}$-subalgebra which is finitely generated as a $\mathbf{Z}$-module and whose $\mathbf{Q}$-span is $\mathfrak{L}(G)$. For almost all primes p, $\zeta_{G,p}(s) = \zeta_{\mathcal{L} \otimes_{\mathbf{Z}} \mathbf{Z}_p}(s)$ and $\zeta_{G,p}^{\triangleleft}(s) = \zeta_{\mathcal{L} \otimes_{\mathbf{Z}} \mathbf{Z}_p}^{\triangleleft}(s)$.*

Constructing Laurent polynomials from algebras and modules. We now recall the explicit description of $\boldsymbol{f}$ in Theorem 2.2 given by du Sautoy and Grunewald; our exposition is equivalent to [27, Rem. 2.7(ii)]. First, choose $\mathfrak{o}$-bases of $\mathcal{A}$ or M in Theorem 2.2 to identify $\mathcal{A} = \mathfrak{o}^d$ or $M = \mathfrak{o}^d$ as $\mathfrak{o}$-modules, respectively. We are then either given a bilinear multiplication $\beta\colon \mathfrak{o}^d \otimes_{\mathfrak{o}} \mathfrak{o}^d \to \mathfrak{o}^d$ turning $\mathfrak{o}^d$ into an $\mathfrak{o}$-algebra or a finite generating set $\mathcal{M}$ of $\mathcal{E} \subset \mathrm{M}_d(\mathfrak{o})$. Let $R := \mathfrak{o}[X_{ij} : 1 \leqslant i \leqslant j \leqslant d]$ and let $C := [X_{ij}]_{i \leqslant j} \in \mathrm{Tr}_d(R)$ with rows $C_1, \ldots, C_d$. We think of C as parameterising a generic $\mathfrak{o}$-submodule of $\mathfrak{o}^d$ via its row span. We extend β to a map $R^d \otimes_R R^d \to R^d$ in the natural way. For Theorem 2.2(i), let $\boldsymbol{f}$ consist of the non-zero entries of $\det(C)^{-1}\beta(C_m, C_n)\,\mathrm{adj}(C)$ for $1 \leqslant m, n \leqslant d$; for part (ii), we instead consider the entries of $\det(C)^{-1}(CM)\,\mathrm{adj}(C)$ as M ranges over $\mathcal{M}$.

3 Toric data

In this section, we introduce the basic object for all of our algorithms: toric data. These objects are closely related to the cone integral data introduced in [13]. In particular, they also give rise to associated p-adic integrals and topological zeta functions.

3.1 Basics

By a **half-open cone** in $\mathbf{R}^n$ we mean a set of the form

$$\mathcal{C}_0 = \big\{\omega \in \mathbf{R}^n : \langle \phi_1, \omega \rangle, \ldots, \langle \phi_d, \omega \rangle \geqslant 0, \quad \langle \chi_1, \omega \rangle, \ldots, \langle \chi_e, \omega \rangle > 0 \big\},$$

where $\langle -, - \rangle$ denotes the standard inner product and $\phi_i, \chi_j \in \mathbf{R}^n$. We say that $\mathcal{C}_0$ is **rational** if we may choose the ϕ_i, χ_j among elements of $\mathbf{Z}^n$.

Definition 3.1. A **toric datum** in n variables over k is a pair $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$ consisting of a half-open rational cone $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$ and a finite family $\boldsymbol{f} = (f_1, \ldots, f_r)$ of Laurent polynomials $f_1, \ldots, f_r \in k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

We often write $(\mathcal{C}_0; f_1, \ldots, f_r)$ instead of $(\mathcal{C}_0; \boldsymbol{f})$. Furthermore, we usually omit the references to n and k. The non-negativity assumption $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$ ensures the convergence of the integrals in §3.2 below. As we will explain in Remark 3.4, toric data provide us with a convenient formalism for describing the domain of integration for the integrals in Theorem 2.2. Recall the notation from p. 4.

Notation 3.2. Given $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$ as in Definition 3.1 and a p-adic field $K \supset k$, we write

$$\mathcal{T}_K := (\mathcal{C}_0; \boldsymbol{f})_K := \Big\{ \boldsymbol{x} \in \mathbf{T}^n(K) : \nu_K(\boldsymbol{x}) \in \mathcal{C}_0, \|\boldsymbol{f}(\boldsymbol{x})\|_K \leqslant 1 \Big\}.$$

Toric data and cone conditions. For an explanation of our terminology, suppose that $\mathcal{C}_0$ is closed. Each of the conditions $\nu_K(\boldsymbol{x}) \in \mathcal{C}_0$ and $\|\boldsymbol{f}(\boldsymbol{x})\|_K \leqslant 1$ can be expressed as a conjunction of finitely many divisibility conditions $v(\boldsymbol{x}) \mid w(\boldsymbol{x})$, where $v, w \in k[\boldsymbol{X}]$. Indeed, let $\mathcal{C}_0 = \{\omega \in \mathbf{R}^n : \langle \phi_1, \omega \rangle, \ldots, \langle \phi_d, \omega \rangle \geqslant 0\}$ for $\phi_1, \ldots, \phi_d \in \mathbf{Z}^n$ and $\boldsymbol{f} = (f_1, \ldots, f_r)$. It is easy to see that $(\mathcal{C}_0; \boldsymbol{f})_K = \big(\mathbf{R}^n_{\geqslant 0}; f_1, \ldots, f_r, \boldsymbol{X}^{\phi_1}, \ldots, \boldsymbol{X}^{\phi_d}\big)_K$. Next,

for a Laurent polynomial $g = \boldsymbol{X}^{-\gamma} g^+$ with $\gamma \in \mathbf{N}_0^n$ and $g^+ \in \mathfrak{O}_K[\boldsymbol{X}]$, the condition $|g(\boldsymbol{x})|_K \leqslant 1$ (where $\boldsymbol{x} \in \mathbf{T}^n(K) \cap \mathfrak{O}_K^n = (\mathbf{R}_{\geqslant 0}^n; \varnothing)_K$) is equivalent to $\boldsymbol{x}^\gamma \mid g^+(\boldsymbol{x})$.

Hence, for $\mathcal{C}_0$ closed, a toric datum gives rise to a special case of a "cone condition" as defined in [13, Def. 1.2(1)]. Moreover, if $\mathcal{D}_K$ denotes the set of K-points of such a cone condition, then $(\mathcal{C}_0; \boldsymbol{f})_K = \mathcal{D}_K \cap \mathbf{T}^n(K)$.

Definition 3.3. A toric datum $(\mathcal{C}_0; \boldsymbol{f})$ is **trivial** if $\mathcal{C}_0 = \varnothing$.

3.2 Zeta functions associated with toric data

Let $\beta \in \mathrm{M}_{m \times n}(\mathbf{N}_0)$ with rows $\beta_1, \ldots, \beta_m$. Given a toric datum $\mathcal{T}$ in n variables over k, a p-adic field $K \supset k$, and $s_1, \ldots, s_m \in \mathbf{C}$ with $\mathrm{Re}(s_j) \geqslant 0$, we consider the zeta function

$$\mathsf{Z}_K^{\mathcal{T},\beta}(s_1, \ldots, s_m) := \int_{\mathcal{T}_K} |\boldsymbol{x}^{\beta_1}|_K^{s_1} \cdots |\boldsymbol{x}^{\beta_m}|_K^{s_m} \, \mathrm{d}\mu_K(\boldsymbol{x}); \tag{3.1}$$

convergence is guaranteed by the non-negativity assumptions $\mathcal{C}_0 \subset \mathbf{R}_{\geqslant 0}^n$ and $\beta_1, \ldots, \beta_m \in \mathbf{N}_0^n$. We note that $\mathsf{Z}_K^{\mathcal{T},\beta}(s_1, \ldots, s_m)$ is a special case of the zeta functions studied in [27, §4].

Remark 3.4 (Local subalgebra and submodule zeta functions)**.** Disregarding factors of the form $(1 - q_K^{-1})^{\pm d}$, Theorem 2.2 shows that zeta functions associated with toric data generalise local zeta functions arising from the enumeration of (normal) subgroups, subalgebras, and submodules as in §2; cf. [27, Rem. 4.12]. Indeed, let $\mathcal{A}$ (or M) in Theorem 2.2 have rank d and let $n = d(d+1)/2$. We identify Tr_d with $\mathbf{A}^n$ via $(x_{ij}) \mapsto (x_{11}, \ldots, x_{1d}, x_{22}, \ldots, x_{dd})$ and let $\boldsymbol{f} \subset \mathfrak{o}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ be an associated family of Laurent polynomials as in Theorem 2.2; see the end of §2 for an explicit construction. Let $\mathcal{T} = (\mathbf{R}_{\geqslant 0}^n; \boldsymbol{f})$ and let $\beta \in \mathrm{M}_{d \times n}(\mathbf{N}_0)$ be the matrix whose jth row corresponds to the elementary matrix with entry 1 in position (j, j) under the above isomorphism $\mathrm{Tr}_d \approx \mathbf{A}^n$. Then $\mathsf{Z}_K^{\mathcal{T},\beta}(s_1, \ldots, s_m)$ specialises to the integral in Theorem 2.2 via $(s_1, \ldots, s_m) \mapsto (s - 1, \ldots, s - d)$.

Relationship with cone integrals. In §3.1, we explained how toric data give rise to cone conditions from [13] (at least when $\mathcal{C}_0$ is closed). In the same spirit, we may regard the integrals in (3.1) as special cases of (multivariate versions of) the cone integrals in [13]; the pair $(\mathcal{T}, \beta)$ takes the place of the cone integral data in [13]. One of the two important special features of (3.1) compared with cone integrals is that we insist on left-hand sides (that is, divisors) in the divisibility conditions describing the domain of integration being monomial (just as they are in Theorem 2.2). By expressing such divisibility conditions in terms of Laurent polynomials as in §3.1, we naturally adopt a toric point of view. This perspective will prove especially useful in combination with the second key feature of the integrals (3.1), namely the presence of a not necessarily closed half-open cone $\mathcal{C}_0$. Focusing exclusively on integrals of the shape (3.1) allows us to develop specialised techniques for manipulating and evaluating them.

Evaluation in theory: explicit formulae. Consider the K-indexed family of zeta functions $\mathsf{Z}_K^{\mathcal{T},\beta}$ defined in terms of a toric datum $\mathcal{T}$ and a matrix β in (3.1). By applying powerful but typically impractical techniques such as resolution of singularities, it can be shown (cf. [8, 12] and [27, Ex. 5.11(vi)]) that there exist a finite set I and, for each $i \in I$, a rational function $W_i(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) \in \mathbf{Q}(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ and a k-variety V_i with the following property: if $K \supset k$ is a p-adic field, then, unless $\mathfrak{p}_K := \mathfrak{o} \cap \mathfrak{P}_K$ belongs to some finite exceptional set (depending on $\mathcal{T}$ only),

$$\mathsf{Z}_K^{\mathcal{T},\beta}(s_1, \ldots, s_m) = \sum_{i \in I} \#\bar{V}_i(\mathfrak{O}_K/\mathfrak{P}_K) \cdot W_i(q_K, q_K^{-s_1}, \ldots, q_K^{-s_m}). \qquad (3.2)$$

We understand the task of computing the $\mathsf{Z}_K^{\mathcal{T},\beta}(s_1, \ldots, s_m)$ to be the explicit construction of V_i and $W_i(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ as in (3.2).

Topological zeta functions. As originally observed by Denef and Loeser [11] for Igusa's local zeta function, given a formula as in (3.2), under additional assumptions regarding the shapes of the $W_i(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ (see §6.1), we may pass to the limit $q_K \to 1$ and obtain the associated topological zeta function

$$\mathsf{Z}_{\mathrm{top}}^{\mathcal{T},\beta}(s_1, \ldots, s_m) := \sum_{i \in I} \chi(V_i(\mathbf{C})) \cdot \lfloor W_i \rfloor(s_1, \ldots, s_m) \in \mathbf{Q}(s_1, \ldots, s_m); \qquad (3.3)$$

here $\chi(V_i(\mathbf{C}))$ denotes the topological Euler characteristic with respect to an arbitrary embedding of k into $\mathbf{C}$ and $\lfloor W_i \rfloor(s_1, \ldots, s_m) \in \mathbf{Q}(s_1, \ldots, s_m)$ is the constant term of $W_i(q_K, q_K^{-s_1}, \ldots, q_K^{-s_m})$, formally expanded as a series in $q_K - 1$. A particularly noteworthy consequence of [11, (2.4)] (see Theorem 6.2) is that the right-hand side of (3.3) is independent of the choice of the family $\big(V_i, W_i(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)\big)_{i \in I}$ in (3.2).

Evaluation in practice: non-degeneracy. Let $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$ be a toric datum over k. If $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$ in the sense of [27, Def. 4.2(i)], then [27, Thm 4.10] yields an effective version of (3.2) in the sense that it provides *explicit* descriptions of varieties V_i and rational functions $W_i(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ in terms of convex-geometric data associated with various cones and polytopes attached to $\mathcal{T}$ and β (see Theorem 5.8). While further computations involving these objects might still be expensive or even infeasible in large dimensions, they are much more useful than formulae obtained using general resolution algorithms; the latter are usually only practical for $n \leqslant 3$. Recall from Remark 3.4 that for the computation of subalgebra or submodule zeta functions, $n = d(d+1)/2$, where d is the additive rank of the object under consideration.

If $\boldsymbol{f}$ satisfies the generally stronger condition of being globally non-degenerate (see [27, Def. 4.2(ii)]), then the Euler characteristics in (3.3) can be expressed in terms of mixed volumes via the Bernstein-Kushnirenko-Khovanskii Theorem, yielding an effective form of (3.3) (see [27, Thm 6.7]). The theory underpinning Algorithm 4.1 in §4 (to be developed in the present article) draws upon and extends this result to overcome certain instances of degeneracy.

Topological subalgebra and submodule zeta functions. Under rather weak technical assumptions on the $W_i(\boldsymbol{q}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ appearing in (3.3), passing from local to topological zeta functions commutes with affine specialisations of the variables $s_1, \dots, s_m$ (see [27, Rem. 5.15]). In particular, we obtain univariate versions of (3.2) and (3.3) arising from the integrals in Theorem 2.2 and thus rigorous definitions of topological subalgebra and submodule zeta functions; see [27, Def. 5.17] or §6.1 below.

4 The main algorithm

In this section, we give a high-level description of an algorithm which seeks to compute topological zeta functions associated with toric data. Our main application is the computation of topological subalgebra and submodule zeta functions via the univariate specialisations explained in Remark 3.4.

The algorithm. We suppose that we are given an initial toric datum $\mathcal{T}^0$, say $\mathcal{T}^0 = (\mathcal{C}_0; \boldsymbol{f})$, in n variables over k—in practice, we are primarily interested in the case where $\boldsymbol{f}$ is a family of Laurent polynomials arising from Theorem 2.2 and $\mathcal{C}_0 = \mathbf{R}_{\geqslant 0}^n$ (where $n = d(d+1)/2$ as in Remark 3.4. Given $\mathcal{T}^0$ and $\beta \in \mathrm{M}_{m \times n}(\mathbf{N}_0)$, the function TOPOLOGICALZETAFUNCTION (Algorithm 4.1) attempts to compute the topological zeta function $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T}^0, \beta}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m) \in \mathbf{Q}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)$ associated with the p-adic integrals $\mathsf{Z}_K^{\mathcal{T}^0, \beta}(s_1, \dots, s_m)$ in (3.1); see §6.2 for a rigorous definition of $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T}^0, \beta}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)$. We note that from now on, we use bold face letters $\boldsymbol{s}_j$ to distinguish variables over $\mathbf{Q}$ from the complex numbers s_j in §3.2.

The computation of $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T}^0, \beta}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)$ using TOPOLOGICALZETAFUNCTION proceeds in two stages. If successful, the first stage uses the functions SIMPLIFY, BALANCE, and REDUCE to construct a family of toric data derived from $\mathcal{T}^0$. In the second stage, the topological zeta functions associated with these derived toric data are computed using the function EVALUATETOPOLOGICALLY; the sum of all these topological zeta functions then gives the final result.

We now explain both stages of TOPOLOGICALZETAFUNCTION in greater detail and we also discuss the roles played by the functions SIMPLIFY, BALANCE, REDUCE, and EVALUATETOPOLOGICALLY. Further details will be given in the following sections.

Stage I: the main loop. During the first stage (lines 1–13) of TOPOLOGICALZETA-FUNCTION, we maintain two lists, unprocessed and regular, of toric data. The essential point here is that unless the execution of TOPOLOGICALZETAFUNCTION is aborted in line 12, each iteration of the loop in lines 2–13 preserves the following property:

($\clubsuit$) There exists a finite $S \subset \mathrm{Spec}(\mathfrak{o})$ such that if $K \supset k$ is a p-adic field with $\mathfrak{p}_K \notin S$, then $\mathcal{T}_K^0$ is the disjoint union of all $\mathcal{T}_K$ with $\mathcal{T}$ ranging over unprocessed $\sqcup$ regular.

In particular, always $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T}^0, \beta}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m) = \sum_{\mathcal{T}} \mathsf{Z}_{\mathrm{top}}^{\mathcal{T}, \beta}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)$, where $\mathcal{T}$ again ranges over unprocessed $\sqcup$ regular (Lemma 6.7).

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Algorithm 4.1 TOPOLOGICALZETAFUNCTION($\mathcal{T}^0, \beta$)

Input: a toric datum $\mathcal{T}^0$ in n variables over k, a matrix $\beta \in \mathrm{M}_{m \times n}(\mathbf{N}_0)$

Output: the topological zeta function $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T}^0, \beta}(s_1, \ldots, s_m) \in \mathbf{Q}(s_1, \ldots, s_m)$ or `fail`

  1: unprocessed $\leftarrow [\mathcal{T}^0]$, regular $\leftarrow [\,]$,   ▷ Stage I

  2: **while** unprocessed is non-empty **do**

  3:      remove an element $\mathcal{T}$ from unprocessed

  4:      $\mathcal{T} \leftarrow$ SIMPLIFY($\mathcal{T}$)

  5:      **if** $\mathcal{T}$ is not balanced **then**

  6:         new $\leftarrow$ BALANCE($\mathcal{T}$)

  7:      **else if** $\mathcal{T}$ is regular **then**

  8:         add $\mathcal{T}$ to regular

  9:         new $\leftarrow [\,]$

10:      **else**

11:         new $\leftarrow$ REDUCE($\mathcal{T}$)

12:         **if** new $=$ `fail` **then return** `fail`

13:      add the non-trivial elements of new to unprocessed

14: **return** $\sum\limits_{\mathcal{T} \in \mathrm{regular}}$ EVALUATETOPOLOGICALLY($\mathcal{T}, \beta$)   ▷ Stage II

The central notions, to be defined in §5, featuring in the while-loop in Algorithm 4.1 are those of *balanced* and *regular* toric data. The function BALANCE (see §5.2) takes as input an arbitrary toric datum and returns a distinguished family of associated balanced ones. Among balanced toric data, regular ones constitute a subclass which is intimately related to the concept of non-degeneracy used in [27]. Namely, given a toric datum $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$, the family $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$ in the sense of [27, Def. 4.2(i)] if and only if each element of BALANCE($\mathcal{T}$) is regular (Proposition 5.7). Our objective during the first stage of Algorithm 4.1 is to successively modify and decompose toric data until, hopefully, at some point all toric data under consideration will be regular. In the reduction step (lines 11–12), given a balanced toric datum $\mathcal{T}$ that fails to be regular, the function REDUCE (see §7.3) attempts to mitigate this failure of regularity by allowing us to replace $\mathcal{T}$ by a suitable family of toric data. This step is supported by the function SIMPLIFY (see §7.2) which removes certain redundancies from its input.

Stage II: topological evaluation. After successful termination of the while-loop in Algorithm 4.1, the computation of the topological zeta function $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T}^0, \beta}(s_1, \ldots, s_m)$ is reduced to computing the topological zeta functions associated with a (possibly large) number of regular toric data via the function EVALUATETOPOLOGICALLY (see §6.7). Given a regular toric datum, by the aforementioned connection between regularity and non-degeneracy and as previously indicated in §3.2, [27] provides us with explicit convex-geometric p-adic formulae. Using [27, §6], we may then deduce a formula for $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T}^0, \beta}(s_1, \ldots, s_m)$ in terms of the topological Euler characteristics of certain subvarieties of tori. However, in contrast to the assumption of global non-degeneracy in [27, Thm 6.7], regularity alone does not suffice to provide us with convex-geometric formulae for these Euler characteristics; as we will explain in §6.6, these Euler characteristics can, however, always be computed.

Remark 4.1.

(i) Write $(\mathcal{D}_0; \boldsymbol{g}) := \text{SIMPLIFY}(\mathcal{T}^0)$. If $\boldsymbol{g}$ is non-degenerate relative to $\mathcal{D}_0$ in the sense of [27, Def. 4.2(i)], then the first stage of TOPOLOGICALZETAFUNCTION will always succeed (see Remark 6.15(ii)). If $\boldsymbol{g}$ is globally non-degenerate (see [27, Def. 4.2(ii)]), then TOPOLOGICALZETAFUNCTION constitutes an algorithmic version of [27, Thm 6.7] for the class of topological zeta functions considered here. However, TOPOLOGICALZETAFUNCTION can do much more: the reduction (lines 11–12) and simplification (line 4) steps allow it to overcome certain instances of degeneracy. When it comes to the practical computation of topological subalgebra and submodule zeta functions, the computations documented in §9 demonstrate that TOPOLOGICALZETAFUNCTION substantially extends the scope of [27]. It is however important to note that a possible point of failure remains in TOPOLOGICALZETAFUNCTION, namely as part of the reduction step in line 12.

(ii) The separation of TOPOLOGICALZETAFUNCTION into two stages might seem artificial. It is justified by the aforementioned chance of failure of TOPOLOGICALZETAFUNCTION and the observation that the final evaluation in line 14 is often the most computationally expensive step; see the final paragraph of §9.1 for an example.

(iii) The first stage of TOPOLOGICALZETAFUNCTION is p-adic in nature in the sense that it consists entirely of manipulations of p-adic integrals. In contrast, the second stage is inherently topological. Finding a practically useful p-adic version of the second stage and thus a practical method for computing associated local zeta functions is a natural direction for future research.

5 Balanced and regular toric data

5.1 Background: cones, polytopes, and polynomials

The following summary of well-known material is based upon [27, §§3.1, 3.3, 4.1].

Cones. A (closed) **cone** in $\mathbf{R}^n$ is a set of the form $\mathcal{C} = \big\{ \omega \in \mathbf{R}^n : \langle \phi_1, \omega \rangle, \ldots, \langle \phi_d, \omega \rangle \geqslant 0 \big\}$, where $\phi_1, \ldots, \phi_d \in \mathbf{R}^n$. Equivalently, cones in $\mathbf{R}^n$ are sets of the form

$$\text{cone}(\varrho_1, \ldots, \varrho_r) := \mathbf{R}_{\geqslant 0} \cdot \varrho_1 + \cdots + \mathbf{R}_{\geqslant 0} \cdot \varrho_r,$$

where $\varrho_1, \ldots, \varrho_r \in \mathbf{R}^n$; we call the $\varrho_1, \ldots, \varrho_r$ a system of **generators** of $\text{cone}(\varrho_1, \ldots, \varrho_r)$. A half-open cone as defined in §3.1 is precisely a set of the form $\mathcal{C} \setminus (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r)$, where $\mathcal{C}$ is a cone, $r \geqslant 0$, and $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are faces of $\mathcal{C}$. In particular, cones are half-open cones. The **relative interior** $\text{relint}(M)$ of a subset $M \subset \mathbf{R}^n$ is the interior of M within its affine hull; see [26, §6] for basic facts. If $M = \text{relint}(M)$, then M is **relatively open**. As a special case of a half-open cone, by a **relatively open cone**, we mean the relative interior of a cone.

Polytopes and normal cones. By a **polytope**, we mean the convex hull of a finite set within some $\mathbf{R}^n$. We insist that faces of a polytope be non-empty. For a non-empty polytope $\mathcal{P} \subset \mathbf{R}^n$ and $\omega \in \mathbf{R}^n$, let $\mathrm{face}_\omega(\mathcal{P})$ denote the face of $\mathcal{P}$ where $\langle -, \omega \rangle$ attains its minimum; note that $\mathrm{face}_0(\mathcal{P}) = \mathcal{P}$. If $\mathcal{Q} \subset \mathbf{R}^n$ is another non-empty polytope, then $\mathrm{face}_\omega(\mathcal{P} + \mathcal{Q}) = \mathrm{face}_\omega(\mathcal{P}) + \mathrm{face}_\omega(\mathcal{Q})$. The **dual cone** $\mathcal{P}^* := \{\omega \in \mathbf{R}^n : \langle \alpha, \omega \rangle \geqslant 0 \text{ for all } \alpha \in \mathcal{P}\}$ of $\mathcal{P}$ is a cone. The **normal cone** of a face $\tau \subset \mathcal{P}$ is the relatively open cone $\mathrm{N}_\tau(\mathcal{P}) := \{\omega \in \mathbf{R}^n : \mathrm{face}_\omega(\mathcal{P}) = \tau\}$ of dimension $n - \dim(\tau)$. The $\mathrm{N}_\tau(\mathcal{P})$ partition $\mathbf{R}^n$ into relatively open cones. The normal cones of faces of $\mathcal{P} + \mathcal{Q}$ coincide with the non-empty sets of the form $\mathrm{N}_\tau(\mathcal{P}) \cap \mathrm{N}_v(\mathcal{Q})$ for faces $\tau \subset \mathcal{P}$ and $v \subset \mathcal{Q}$, corresponding to the unique decomposition of a face of $\mathcal{P} + \mathcal{Q}$ into a sum $\tau + v$.

Newton polytopes and initial forms. Let $f = \sum_{\alpha \in \mathbf{Z}^n} c_\alpha \mathbf{X}^\alpha \in k[\mathbf{X}^{\pm 1}]$, where $c_\alpha \in k$. The **support** of f is $\mathrm{supp}(f) := \{\alpha \in \mathbf{Z}^n : c_\alpha \neq 0\}$. The **Newton polytope** $\mathrm{New}(f)$ of f is the convex hull of $\mathrm{supp}(f)$ within $\mathbf{R}^n$. If $f, g \in k[\mathbf{X}^{\pm 1}]$, then $\mathrm{New}(fg) = \mathrm{New}(f) + \mathrm{New}(g)$. For $\omega \in \mathbf{R}^n$, the **initial form** $\mathrm{in}_\omega(f)$ of f in the direction ω is the sum of those $c_\alpha \mathbf{X}^\alpha$ with $\alpha \in \mathrm{supp}(f)$ where $\langle \alpha, \omega \rangle$ attains its minimum. If $f \neq 0$, then $\mathrm{face}_\omega(\mathrm{New}(f)) = \mathrm{New}(\mathrm{in}_\omega(f))$ for $\omega \in \mathbf{R}^n$. The equivalence classes on $\mathbf{R}^n$ of $\omega \sim \omega' :\iff \mathrm{in}_\omega(f) = \mathrm{in}_{\omega'}(f)$ are the normal cones of the faces of $\mathrm{New}(f)$.

5.2 Balanced toric data

Definition 5.1.

(i) Let $\mathcal{C}_0 \subset \mathbf{R}^n$ be arbitrary. We say that $f \in k[\mathbf{X}^{\pm 1}]$ is $\mathcal{C}_0$-**balanced** if $\mathrm{in}_\omega(f)$ is constant as ω ranges over $\mathcal{C}_0$. If $\mathcal{C}_0 \neq \varnothing$, we denote this common initial form by $\mathrm{in}_{\mathcal{C}_0}(f)$.

(ii) A toric datum $(\mathcal{C}_0; f_1, \ldots, f_r)$ is **balanced** if each f_i is $\mathcal{C}_0$-balanced.

Example 5.2. Let $n = 2$ and k be arbitrary. Define $f_1 = X_1^{-1} - X_2^{-1}$ and $f_2 = X_1^{-2} - X_2^{-2}$. Then $(\mathbf{R}_{\geqslant 0}^2; f_1, f_2)$ is not balanced. Indeed, $\mathrm{supp}(f_1) = \{(-1, 0), (0, -1)\}$,

$$-1 = \langle (-1, 0), (1, 0) \rangle < \langle (0, -1), (1, 0) \rangle = 0,$$

and

$$-1 = \langle (0, -1), (0, 1) \rangle < \langle (-1, 0), (0, 1) \rangle = 0$$

whence $\mathrm{in}_{(1,0)}(f_1) = X_1^{-1}$ and $\mathrm{in}_{(0,1)}(f_1) = -X_2^{-1}$. Let $\mathcal{C}_0 = \{(\omega_1, \omega_2) \in \mathbf{R}_{\geqslant 0}^2 : \omega_1 > \omega_2\}$. Then $(\mathcal{C}_0; f_1, f_2)$ is balanced with $\mathrm{in}_{\mathcal{C}_0}(f_1) = X_1^{-1}$ and $\mathrm{in}_{\mathcal{C}_0}(f_2) = X_1^{-2}$.

Lemma 5.3. *Let $\mathcal{T} = (\mathcal{C}_0; f_1, \ldots, f_r)$ be a toric datum with $f_1 \cdots f_r \neq 0$. Let $\mathcal{N} = \mathrm{New}(f_1 \cdots f_r)$. Then $\mathcal{T}$ is balanced if and only if $\mathcal{C}_0 \subset \mathrm{N}_\tau(\mathcal{N})$ for some face $\tau \subset \mathcal{N}$.*

Proof. We freely use the content of §5.1. We may assume that $\mathcal{C}_0 \neq \varnothing$. Suppose that $\mathcal{T}$ is balanced. For $1 \leqslant i \leqslant r$, since f_i is $\mathcal{C}_0$-balanced, there exists a face $\tau_i \subset \mathrm{New}(f_i)$ with $\mathcal{C}_0 \subset \mathrm{N}_{\tau_i}(\mathrm{New}(f_i))$. Hence, $\varnothing \neq \mathcal{C}_0 \subset \bigcap_{i=1}^r \mathrm{N}_{\tau_i}(\mathrm{New}(f_i))$ and $\tau := \sum_{i=1}^r \tau_i$ is the desired face of $\sum_{i=1}^r \mathrm{New}(f_i) = \mathcal{N}$. Conversely, let $\mathcal{C}_0 \subset \mathrm{N}_\tau(\mathcal{N})$ and write $\tau = \sum_{i=1}^r \tau_i$

for faces $\tau_i \subset \mathrm{New}(f_i)$. Then $\mathcal{C}_0 \subset \mathrm{N}_\tau(\mathcal{N}) \subset \mathrm{N}_{\tau_i}(\mathrm{New}(f_i))$ for $1 \leqslant i \leqslant r$, whence $\mathcal{T}$ is balanced. $\qquad\qquad\blacklozenge$

The following notion will be used to show that TOPOLOGICALZETAFUNCTION (Algorithm 4.1) preserves ($\clubsuit$) from p. 9.

Definition 5.4. Let $\mathcal{T}$ be a toric datum. A finite collection $\mathfrak{C}$ of toric data is a **partition** of $\mathcal{T}$ if there exists a finite $S \subset \mathrm{Spec}(\mathfrak{o})$ such that if $K \supset k$ is a p-adic field with $\mathfrak{p}_K \notin S$, then $\mathcal{T}_K = \bigcup_{\mathcal{T}' \in \mathfrak{C}} \mathcal{T}'_K$ (disjoint).

We may thus rephrase ($\clubsuit$) by stating that unprocessed $\sqcup$ regular is a partition of $\mathcal{T}^0$.

The function BALANCE. Given a toric datum $(\mathcal{C}_0; f_1, \ldots, f_r)$, the function BALANCE produces a partition consisting of balanced toric data as follows. Let $I = \{i : f_i \neq 0\}$ and $\mathcal{N} = \mathrm{New}(\prod_{i \in I} f_i)$. Let $\mathrm{BALANCE}(\mathcal{C}_0; f_1, \ldots, f_r)$ return the collection of $(\mathcal{C}_0 \cap \mathrm{N}_\tau(\mathcal{N}); f_1, \ldots, f_r)$ for faces $\tau \subset \mathcal{N}$ with $\mathcal{C}_0 \cap \mathrm{N}_\tau(\mathcal{N}) \neq \varnothing$. Note that each toric datum $(\mathcal{C}_0 \cap \mathrm{N}_\tau(\mathcal{N}); f_1, \ldots, f_r)$ is balanced by Lemma 5.3.

5.3 Regular toric data

Let $\bar{k}$ be an algebraic closure of k.

Definition 5.5. A balanced toric datum $(\mathcal{C}_0; f_1, \ldots, f_r)$ is **regular** if either $\mathcal{C}_0 = \varnothing$ or the following condition is satisfied:

> For all $J \subset \{1, \ldots, r\}$, if $\boldsymbol{u} \in \mathbf{T}^n(\bar{k})$ satisfies $\mathrm{in}_{\mathcal{C}_0}(f_j)(\boldsymbol{u}) = 0$ for all $j \in J$, then the Jacobian matrix $\left[\frac{\partial \mathrm{in}_{\mathcal{C}_0}(f_j)}{\partial X_i}(\boldsymbol{u})\right]_{i=1,\ldots,n; j \in J}$ has rank $|J|$.

We say that $(\mathcal{C}_0; f_1, \ldots, f_r)$ is **singular** if it is balanced but not regular.

Example 5.6. The toric datum $(\mathcal{C}_0; f_1, f_2)$ in Example 5.2 is trivially regular since both initial forms $\mathrm{in}_{\mathcal{C}_0}(f_1)$ and $\mathrm{in}_{\mathcal{C}_0}(f_2)$ are Laurent monomials and hence do not vanish on $\mathbf{T}^2(\bar{k})$. Let $\mathcal{C}'_0 = \{(\omega_1, \omega_2) \in \mathbf{R}^2_{\geqslant 0} : \omega_1 = \omega_2\}$. Then $(\mathcal{C}'_0; f_1, f_2)$ is balanced but singular. Indeed, the initial forms are $\mathrm{in}_{\mathcal{C}'_0}(f_1) = f_1$ and $\mathrm{in}_{\mathcal{C}'_0}(f_2) = f_2$ and the condition in Definition 5.5 is violated on the subvariety of $\mathbf{T}^2_k$ defined by $X_1 = X_2$ for $J = \{1, 2\}$.

Regularity is closely related to the notion of non-degeneracy used in [27]. First, by definition, a balanced toric datum $(\mathcal{C}_0; \boldsymbol{f})$ with $0 \notin \boldsymbol{f}$ is regular if and only if $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$ in the sense of [27, Def. 4.2(i)]. More generally, we may rephrase non-degeneracy of Laurent polynomials in terms of regularity of toric data as follows.

Lemma 5.7. *Given $(\mathcal{C}_0; \boldsymbol{f})$ with $0 \notin \boldsymbol{f}$, the family $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$ in the sense of [27, Def. 4.2(i)] if and only if each element of $\mathrm{BALANCE}(\mathcal{C}_0; \boldsymbol{f})$ is regular.* $\qquad\blacklozenge$

Testing regularity. As a part of TOPOLOGICALZETAFUNCTION (Algorithm 4.1), we need to test regularity of toric data. This can be carried out using Gröbner bases computations as follows. Let $(\mathcal{C}_0; f_1, \ldots, f_r)$ be a balanced toric datum with $\mathcal{C}_0 \neq \varnothing$. Write $g_i := \mathrm{in}_{\mathcal{C}_0}(f_i)$ and $M_J := \left[\frac{\partial g_j}{\partial X_i}\right]_{i=1,\ldots,n; j \in J}$. By the weak Nullstellensatz, $(\mathcal{C}_0; \boldsymbol{f})$ is regular if and only if for each $J \subset \{1, \ldots, r\}$, the Laurent polynomials g_j for $j \in J$ together with the $|J| \times |J|$-minors of M_J generate the unit ideal of $k[\boldsymbol{X}^{\pm 1}]$. For practical computations, it is convenient to express the preceding characterisation of regularity of $(\mathcal{C}_0; \boldsymbol{f})$ in terms of ideals of the polynomial algebra $k[\boldsymbol{X}]$. Thus, since regularity of $(\mathcal{C}_0; f_1, \ldots, f_r)$ is invariant under rescaling of the f_i by Laurent monomials (cf. [27, Rem. 4.3(ii)]), we may assume that $g_1, \ldots, g_r \in k[\boldsymbol{X}]$ are polynomials. It follows that $(\mathcal{C}_0; f_1, \ldots, f_r)$ is regular if and only if, for all J, the monomial $X_1 \cdots X_n$ is contained in the radical of the ideal generated by all g_j ($j \in J$) and the $|J| \times |J|$-minors of M_J within $k[\boldsymbol{X}]$. Using the Rabinowitsch trick (see e.g. the proof of [23, Thm IX.1.5]), the latter condition can now be tested using Gröbner bases machinery.

5.4 Reminder: generating functions of cones

The following material is largely well-known; see e.g. [4, Ch. 13] and [30, §4.5].

Generating functions. Given a cone $\mathcal{C} \subset \mathbf{R}^n$ and a ring R, let $R[\mathcal{C} \cap \mathbf{Z}^n]$ be the R-subalgebra of $R[\boldsymbol{X}^{\pm 1}]$ spanned by $\boldsymbol{X}^\alpha$ with $\alpha \in \mathcal{C} \cap \mathbf{Z}^n$. If $\mathcal{C} \subset \mathbf{R}_{\geqslant 0}^n$ is a *rational* cone, then, within the field of fractions of $\mathbf{Q}[\![\boldsymbol{X}]\!]$, the generating function $\sum_{\omega \in \mathcal{C} \cap \mathbf{N}_0^n} \boldsymbol{X}^\omega \in \mathbf{Q}[\![\boldsymbol{X}]\!]$ is given by a rational function $|\mathcal{C}|$ of the form $|\mathcal{C}| = f(\boldsymbol{X})/\prod_{i=1}^r (1 - \boldsymbol{X}^{\alpha_i})$, where $f(\boldsymbol{X}) \in \mathbf{Z}[\mathcal{C} \cap \mathbf{Z}^n]$ and $\alpha_1, \ldots, \alpha_r \in \mathcal{C} \cap \mathbf{N}_0^n$. For an analytic characterisation of $|\mathcal{C}|$, let $\mathcal{C} = \mathrm{cone}(\varrho_1, \ldots, \varrho_e)$ for $0 \neq \varrho_j \in \mathbf{N}_0^n$. Then $\mathrm{U}(\mathcal{C}) := \{\boldsymbol{x} \in \mathbf{T}^n(\mathbf{C}) : |\boldsymbol{x}^{\varrho_j}| < 1 \text{ for } j = 1, \ldots, e\}$ is a non-empty open set which is independent of the choice of $\varrho_1, \ldots, \varrho_e$, and $|\mathcal{C}|(\boldsymbol{x}) = \sum_{\omega \in \mathcal{C} \cap \mathbf{N}_0^n} \boldsymbol{x}^\omega$ for all $\boldsymbol{x} \in \mathrm{U}(\mathcal{C})$, the convergence being absolute and compact on $\mathrm{U}(\mathcal{C})$.

Triangulation. The function $|\mathcal{C}|$ can be computed in terms of a triangulation of $\mathcal{C}$. Here, by a **triangulation** of $\mathcal{C}$, we mean a rational polyhedral fan $\mathcal{F}$ in $\mathbf{R}^n$ which consists of simplicial cones and whose support is $\mathcal{C}$; for the existence of a triangulation of $\mathcal{C}$, see [7, Prop. 11.1.7] and cf. [30, Lem. 4.5.1]. If $\sigma, \sigma' \in \mathcal{F}$ are distinct cones of the same dimension, d say, then $\sigma \cap \sigma' \in \mathcal{F}$ is a proper face of σ or of σ' whence $\dim(\sigma \cap \sigma') < d$. Using the inclusion-exclusion principle as in [4, Lem. 13.7], we may write $|\mathcal{C}|$ as a $\mathbf{Z}$-linear combination of the rational functions $|\sigma|$ for $\sigma \in \mathcal{F}$; it follows from the preceding remark that those $\sigma \in \mathcal{F}$ with $\dim(\sigma) = \dim(\mathcal{C})$ have coefficient 1. If σ is simplicial, say $\sigma = \mathrm{cone}(\beta_1, \ldots, \beta_d)$ with $d = \dim(\sigma)$, then $|\sigma| = (\sum \boldsymbol{X}^\alpha)/\prod_{i=1}^d (1 - \boldsymbol{X}^{\beta_i})$, where the summation in the numerator extends over the lattice points in the half-open parallelepiped $\{\sum_{i=1}^d a_i \beta_i : a_i \in \mathbf{R}, 0 \leqslant a_i < 1\}$; see [4, Lem. 13.5]. Using the inclusion-exclusion principle again, what has been said about closed cones above extends to (rational) half-open cones $\mathcal{C}_0 \subset \mathbf{R}_{\geqslant 0}^n$ (see [27, §3.1]); in particular, we obtain a rational function $|\mathcal{C}_0|$ enumerating the lattice points in $\mathcal{C}_0$.

5.5 Local zeta functions associated with regular toric data

We record how the machinery developed in [27] provides convex-geometric formulae for local zeta functions associated with regular toric data.

Monomial substitutions. (Cf. [27, §3.2].) Let $A \in \mathrm{M}_{n \times (m+1)}(\mathbf{N}_0)$. We assume that the first column of A is $(1, \ldots, 1)^\top$. Write $A_1, \ldots, A_n$ for the rows of A. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_0, \ldots, Y_m)$ consist of independent variables over $\mathbf{Q}$; note the different indices used. Let $\mathcal{A}$ be the $\mathbf{Q}$-subalgebra of $\mathbf{Q}(\boldsymbol{X})$ generated by $\mathbf{Q}[\boldsymbol{X}]$ and all $(1 - \boldsymbol{X}^\alpha)^{-1}$ for $0 \neq \alpha \in \mathbf{N}_0^n$; similarly, let $\mathcal{B}$ be the $\mathbf{Q}$-algebra generated by $\mathbf{Q}[\boldsymbol{Y}]$ and all $(1 - \boldsymbol{Y}^\beta)^{-1}$ for $0 \neq \beta \in \mathbf{N}_0^{m+1}$. Then $\boldsymbol{X}^\alpha \mapsto \boldsymbol{Y}^{\alpha A}$ extends to a homomorphism $(-)^A \colon \mathcal{A} \to \mathcal{A}'$. In particular, if $\mathcal{C}_0 \subset \mathbf{R}_{\geqslant 0}^n$ is a rational half-open cone, then $|\mathcal{C}_0|$ belongs to $\mathcal{A}$ and we may thus consider its image $|\mathcal{C}_0|^A$. Observe that the rational function $|\mathcal{C}_0|^A$ can be evaluated at any point $(y_0, \ldots, y_m) \in \mathbf{C}^{m+1}$ with $0 \leqslant |y_0| < 1$ and $0 \leqslant |y_j| \leqslant 1$ for $j = 1, \ldots, m$.

Setup. Let $(\mathcal{C}_0; f_1, \ldots, f_r)$ be a regular toric datum over k. We assume that $\mathcal{C}_0 \neq \varnothing$. For $1 \leqslant i \leqslant r$, we choose an arbitrary $\gamma_i \in \mathrm{supp}(\mathrm{in}_{\mathcal{C}_0}(f_i))$. Let $J \subset \{1, \ldots, r\}$. Write $\ell := |J|$ and $J = \{j_1, \ldots, j_\ell\}$ for $j_1 < j_2 < \cdots < j_\ell$. Define V_J° to be the subvariety of $\mathbf{T}_k^n$ defined by the vanishing of all $\mathrm{in}_{\mathcal{C}_0}(f_j)$ for $j \in J$ and the non-vanishing of all remaining $\mathrm{in}_{\mathcal{C}_0}(f_i)$. For $j \in J$, say $j = j_a$, let δ_{jJ} be the ath standard basis vector of $\mathbf{R}^{|J|}$; for $1 \leqslant i \leqslant r$, $i \notin J$, we let $\delta_{iJ} = 0_{\mathbf{R}^{|J|}}$. Define $\mathcal{D}_J$ to be the cone consisting of those $(\xi, o) \in \mathbf{R}_{\geqslant 0}^n \times \mathbf{R}_{\geqslant 0}^{|J|}$ with $\langle \gamma_i, \xi \rangle + \langle \delta_{iJ}, o \rangle \geqslant 0$ for $i = 1, \ldots, r$. Let $\mathcal{C}_0^J := (\mathcal{C}_0 \times \mathbf{R}_{>0}^{|J|}) \cap \mathcal{D}_J$. Finally, for $\beta \in \mathrm{M}_{m \times n}(\mathbf{N}_0)$ with rows $\beta_1, \ldots, \beta_m$, define an $(n + |J|) \times (m + 1)$ matrix $A_J(\beta) := [(1, \ldots, 1)^\top, (\beta_1, 0_{\mathbf{R}^{|J|}})^\top, \ldots, (\beta_m, 0_{\mathbf{R}^{|J|}})^\top]$.

Theorem 5.8 (Cf. [27, Thm 4.10]). *Let $\mathcal{T} = (\mathcal{C}_0; f_1, \ldots, f_r)$ be a regular toric datum over k. Let $\beta \in \mathrm{M}_{m \times n}(\mathbf{N}_0)$. Define V_J°, $\mathcal{C}_0^J$, and $A_J(\beta)$ as above. For all p-adic fields $K \supset k$ and $s_1, \ldots, s_m \in \mathbf{C}$ with $\mathrm{Re}(s_j) \geqslant 0$, unless $\mathfrak{p}_K$ belongs to some finite exceptional set (depending only on $\mathcal{T}$),*

$$\mathsf{Z}_K^{\mathcal{T}, \beta}(s_1, \ldots, s_m) = \sum_{J \subset \{1, \ldots, r\}} \# \bar{V}_J^\circ(\mathfrak{O}_K/\mathfrak{P}_K) \cdot \frac{(q_K - 1)^{|J|}}{q_K^n} \cdot |\mathcal{C}_0^J|^{A_J(\beta)}(q_K^{-1}, q_K^{-s_1}, \ldots, q_K^{-s_m}).$$

6 Topological zeta functions and regular toric data

We describe the function EVALUATETOPOLOGICALLY (see §6.7) which computes the topological zeta function $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T}, \beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ associated with a regular toric datum $\mathcal{T}$ and a matrix β as defined in §6.2. Our method is based on refined and algorithmic versions of the key ingredients of [27, Thm 6.7] applied to Theorem 5.8, with further extensions removing the assumptions of "global non-degeneracy" from [27, §6].

6.1 Reminder: topological zeta functions via p-adic formulae

In §3.2, we sketched how formulae of the form (3.2) can be used to read off the associated topological zeta function (3.3). For a rigorous treatment, we now recall the formalism

of [27] which is based on work of Denef and Loeser [11]. In particular, we recall the technical conditions regarding the W_i in (3.3) alluded to above.

Formal binomial expansions. Let $q, t_1, \ldots, t_m, s_1, \ldots, s_m$ be algebraically independent variables over $\mathbf{Q}$; we regard these variables as symbolic versions of the $q_K, q_K^{-s_1}, \ldots,$ $q_K^{-s_m}, s_1, \ldots, s_m$ in Theorem 5.8. Using the binomial series, we define

$$q^{-s_j} := \sum_{d=0}^{\infty} \binom{-s_j}{d} (q-1)^d \in \mathbf{Q}[s_j][\![q-1]\!].$$

Let $f(q, t_1, \ldots, t_m) \in \mathbf{Q}[q^{\pm 1}, t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$. Let $r \in \mathbf{N}_0$ and for $1 \leqslant i \leqslant r$, let $(a_i, b_i) \in \mathbf{Z}^{1+m}$ with $(a_i, b_i) \neq 0_{\mathbf{Z}^{1+m}}$. Let $W(q, t_1, \ldots, t_m) \in \mathbf{Q}(q, t_1, \ldots, t_m)$ be of the form

$$W(q, t_1, \ldots, t_m) = \frac{f(q, t_1, \ldots, t_m)}{\prod_{i=1}^{r} (q^{a_i} t^{b_i} - 1)}. \tag{6.1}$$

Given $W(q, t_1, \ldots, t_m)$, we obtain $W(q, q^{-s_1}, \ldots, q^{-s_m}) \in \mathbf{Q}(s_1, \ldots, s_m)(\!(q-1)\!)$.

Notation 6.1.

(i) Let $\mathbf{M}$ be the $\mathbf{Q}$-subalgebra of $\mathbf{Q}(q, t_1, \ldots, t_m)$ consisting of those $W(q, t_1, \ldots, t_m)$ of the form (6.1) with $W(q, q^{-s_1}, \ldots, q^{-s_m}) \in \mathbf{Q}(s_1, \ldots, s_m)[\![q-1]\!]$.

(ii) Given $W(q, t_1, \ldots, t_m) \in \mathbf{M}$, we let $\lfloor W \rfloor(s_1, \ldots, s_m) \in \mathbf{Q}(s_1, \ldots, s_m)$ denote the constant term of $W(q, q^{-s_1}, \ldots, q^{-s_m})$ as a series in $q-1$.

The following generalises topological zeta functions of polynomials introduced in [11].

Theorem 6.2 ([27, §5.3])**.** *Let* $\mathsf{Z} = (\mathsf{Z}_K)$ *be a family of rational functions* $\mathsf{Z}_K(t_1, \ldots, t_m) \in \mathbf{Q}(t_1, \ldots, t_m)$ *indexed by p-adic fields $K \supset k$ (up to k-isomorphism). Suppose that there exists a finite family of k-varieties V_i and $W_i(q, t_1, \ldots, t_m) \in \mathbf{M}$ $(i \in I)$ such that for p-adic fields $K \supset k$, unless $\mathfrak{p}_K$ belongs to some finite set,*

$$\mathsf{Z}_K(t_1, \ldots, t_m) = \sum_{i \in I} \#\bar{V}_i(\mathfrak{O}_K/\mathfrak{P}_K) \cdot W_i(q_K, t_1, \ldots, t_m).$$

Then the **topological zeta function**

$$\mathsf{Z}_{\mathrm{top}}(s_1, \ldots, s_m) := \sum_{i \in I} \chi(V_i(\mathbf{C})) \cdot \lfloor W_i \rfloor(s_1, \ldots, s_m) \in \mathbf{Q}(s_1, \ldots, s_m)$$

associated with Z is independent of the choice of $(V_i, W_i(q, t_1, \ldots, t_m))_{i \in I}$.

When $m = 1$, we write s and t instead of s_1 and t_1. The formulae for local subalgebra and submodule zeta functions in [13] give rise to associated topological zeta functions.

Theorem 6.3 ([27, §5.4])**.**

(i) *Let $\mathcal{A}$ be a possibly non-associative $\mathfrak{o}$-algebra whose underlying $\mathfrak{o}$-module is free of finite rank d. For a p-adic field $K \supset k$, let $\mathsf{Z}_{\mathcal{A},K}(\boldsymbol{t}) \in \mathbf{Q}(\boldsymbol{t})$ be the rational function characterised by*

$$\mathsf{Z}_{\mathcal{A},K}(q_K^{-s}) = (1 - q_K^{-1})^d \cdot \zeta_{\mathcal{A} \otimes_\mathfrak{o} \mathfrak{O}_K}(s).$$

Then $\mathsf{Z}_{\mathcal{A}} := (\mathsf{Z}_{\mathcal{A},K})$ satisfies the assumptions in Theorem 6.2.

The **topological subalgebra zeta function** *of $\mathcal{A}$ is*

$$\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s}) := \mathsf{Z}_{\mathcal{A},\mathrm{top}}(\boldsymbol{s}) \in \mathbf{Q}(\boldsymbol{s}).$$

(ii) *Let M be a free $\mathfrak{o}$-module of rank d and let $\mathcal{E}$ be a unital subalgebra of $\mathrm{End}_\mathfrak{o}(M)$. For a p-adic field $K \supset k$, let $\mathsf{Z}_{\mathcal{E} \curvearrowright M,K}(\boldsymbol{t}) \in \mathbf{Q}(\boldsymbol{t})$ be the rational function with*

$$\mathsf{Z}_{\mathcal{E} \curvearrowright M,K}(q_K^{-s}) = (1 - q_K^{-1})^d \cdot \zeta_{(\mathcal{E} \otimes_\mathfrak{o} \mathfrak{O}_K) \curvearrowright (M \otimes_\mathfrak{o} \mathfrak{O}_K)}(s).$$

Then $\mathsf{Z}_{\mathcal{E} \curvearrowright M} := (\mathsf{Z}_{\mathcal{E} \curvearrowright M,K})$ satisfies the assumptions in Theorem 6.2.

The **topological submodule zeta function** *of $\mathcal{E}$ acting on M is*

$$\zeta_{\mathcal{E} \curvearrowright M,\mathrm{top}}(\boldsymbol{s}) := \mathsf{Z}_{\mathcal{E} \curvearrowright M,\mathrm{top}}(\boldsymbol{s}) \in \mathbf{Q}(\boldsymbol{s}).$$

In view of Definition 2.1(ii), we define the **topological ideal zeta function** of $\mathcal{A}$ to be $\zeta_{\mathcal{A},\mathrm{top}}^{\triangleleft}(\boldsymbol{s}) := \zeta_{\Omega(\mathcal{A}) \curvearrowright \mathcal{A},\mathrm{top}}(\boldsymbol{s})$.

Remark 6.4. Topological subalgebra zeta functions were first defined in greater generality by du Sautoy and Loeser [14]. Their definition of the topological subalgebra zeta function of a **Z**-algebra $\mathcal{A}$ of **Z**-rank d coincides with $d! \cdot \mathsf{Z}_{\mathcal{A},\mathrm{top}}(\boldsymbol{s} + d)$ in our notation; we note that the factor $d!$ (a consequence of [14, Def. 7.2] and the remarks following it) seems to be missing from the examples in [14, §9].

The following simple observation will be useful for our computations (see §8).

Lemma 6.5. *In the setting of Theorem 6.3, the univariate rational functions $\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s}) \in \mathbf{Q}(\boldsymbol{s})$ and $\zeta_{\mathcal{E} \curvearrowright M,\mathrm{top}}(\boldsymbol{s}) \in \mathbf{Q}(\boldsymbol{s})$ both have degree $\leqslant 0$ in $\boldsymbol{s}$.*

Proof. This follows from the explicit formula [14, Prop. 8.4] for topological zeta functions associated with cone integrals—or equivalently, the topological counterpart of [13, Cor. 3.2]. Indeed, these formulae express the topological zeta functions under consideration as **Z**-linear combinations of rational functions of degree $\leqslant 0$ in $\boldsymbol{s}$. ♦

Let $\mathcal{A}$ have $\mathfrak{o}$-rank d. For all examples of topological subalgebra zeta functions known to the author, the degree of $\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s})$ is precisely $-d$; see [27, §8, Conj. I].

Theorem 6.6 ([27, §5.4]; cf. [14, §6])**.** *Let G be a finitely generated torsion-free nilpotent group. Let $\mathfrak{L}(G)$ be the associated Lie $\mathbf{Q}$-algebra under the Mal'cev correspondence. Choose an arbitrary $\mathbf{Z}$-subalgebra $\mathcal{L} \subset \mathfrak{L}(G)$ which is finitely generated as a $\mathbf{Z}$-module and which spans $\mathfrak{L}(G)$ over $\mathbf{Q}$. The* **topological subgroup zeta function** *and* **topological normal subgroup zeta function** *of G are $\zeta_{G,\mathrm{top}}(\boldsymbol{s}) := \zeta_{\mathcal{L},\mathrm{top}}(\boldsymbol{s})$ and $\zeta_{G,\mathrm{top}}^{\triangleleft}(\boldsymbol{s}) := \zeta_{\mathcal{L},\mathrm{top}}^{\triangleleft}(\boldsymbol{s})$, respectively. These definitions do not depend on the choice of $\mathcal{L}$.*

6.2 Topological zeta functions associated with toric data

Let $\mathcal{T}$ be a toric datum in n variables over k and let $\beta \in \mathrm{M}_{m \times n}(\mathbf{N}_0)$. The following is a special case of general results in p-adic integration following Denef's fundamental paper [8]; cf. [27, Rem. 4.7] and [27, Ex. 5.11(vi)]. Thus, for each p-adic field $K \supset k$, the zeta function $\mathsf{Z}_K^{\mathcal{T},\beta}(s_1, \ldots, s_m)$ is rational in $q_K^{-s_1}, \ldots, q_K^{-s_m}$. We may therefore regard each $\mathsf{Z}_K^{\mathcal{T},\beta}(s_1, \ldots, s_m)$ as an element of $\mathbf{Q}(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ via $\boldsymbol{t}_j \mapsto q_K^{-s_j}$. After this identification, the collection of rational functions $\mathsf{Z}^{\mathcal{T},\beta} := (\mathsf{Z}_K^{\mathcal{T},\beta})$ satisfies the assumptions in Theorem 6.2. Consequently, we obtain a topological zeta function $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T},\beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) \in \mathbf{Q}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ associated with $\mathcal{T}$ and β.

Lemma 6.7. *Let $\mathcal{T}$ and β be as above. Let $\mathfrak{C}$ be a partition of $\mathcal{T}$ (see Definition 5.4). Then $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T},\beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = \sum_{\mathcal{T}' \in \mathfrak{C}} \mathsf{Z}_{\mathrm{top}}^{\mathcal{T}',\beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m).$*

Proof. There exists a finite $S \subset \mathrm{Spec}(\mathfrak{o})$ such that for all p-adic fields $K \supset k$ with $\mathfrak{p}_K \notin S$, $\mathsf{Z}_K^{\mathcal{T},\beta}(s_1, \ldots, s_m) = \sum_{\mathcal{T}' \in \mathfrak{C}} \mathsf{Z}_K^{\mathcal{T}',\beta}(s_1, \ldots, s_m)$. Now apply Theorem 6.2 to both sides of the preceding equation. ♦

Hence, if we assume that the invariant (♣) on p. 9 is preserved by both SIMPLIFY and REDUCE and that EVALUATETOPOLOGICALLY$(\mathcal{T}, \beta)$ computes $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T},\beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$, then the correctness of TOPOLOGICALZETAFUNCTION (Algorithm 4.1) follows from Lemma 6.7.

6.3 Torus factors

The group $\mathrm{GL}_n(\mathbf{Z})$ admits a natural right-action on $k[\boldsymbol{X}^{\pm 1}]$ by k-algebra automorphisms via $(\boldsymbol{X}^\alpha)^A := \boldsymbol{X}^{\alpha A}$ for $\alpha \in \mathbf{Z}^n$ and $A \in \mathrm{GL}_n(\mathbf{Z})$. Recall the definition of the Newton polytope $\mathrm{New}(f)$ of a Laurent polynomial $f \in k[\boldsymbol{X}^{\pm 1}]$ from §5.1.

Lemma 6.8 ([27, Lem. 6.1(i)]). *Let $f_1, \ldots, f_r \in k[\boldsymbol{X}^{\pm 1}]$ be non-zero Laurent polynomials. Let $\mathcal{N} = \mathrm{New}(f_1 \cdots f_r)$ and $d = \dim(\mathcal{N})$. For $1 \leqslant i \leqslant r$, choose $\alpha_i \in \mathrm{supp}(f_i)$. Then there exists $A \in \mathrm{GL}_n(\mathbf{Z})$ such that $(\boldsymbol{X}^{-\alpha_i} f_i)^A \in k[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ for $1 \leqslant i \leqslant r$.*

Recall that Theorem 5.8 featured certain explicitly defined subvarieties of algebraic tori over k. The relevance of Lemma 6.8 is due to the following geometric consequence.

Corollary 6.9. *Write $g_i := (\boldsymbol{X}^{-\alpha_i} f_i)^A$ for $1 \leqslant i \leqslant r$. Let V be the subvariety of $\mathbf{T}_k^n$ defined by $f_1 = \ldots = f_r = 0$ and let U be the subvariety of $\mathbf{T}_k^d$ defined by $g_1 = \ldots = g_r = 0$. Then $V \approx_k U \times_{\mathrm{Spec}(k)} \mathbf{T}_k^{n-d}$ (isomorphism of k-varieties).* ♦

The proof of Lemma 6.8 given in [27] easily translates into an algorithm. Indeed, let M be the $\mathbf{Z}$-submodule of $\mathbf{Z}^n$ generated by $\bigcup_{i=1}^r \mathrm{supp}(\boldsymbol{X}^{-\alpha_i} f_i)$. Then M has rank d (see the proof of [27, Lem. 6.1(i)]). Let B be a matrix over $\mathbf{Z}$ (of size $e \times n$, say) whose rows span M over $\mathbf{Z}$. We may find $C \in \mathrm{GL}_e(\mathbf{Z})$ and $A \in \mathrm{GL}_n(\mathbf{Z})$ such that CBA is in Smith normal form. Evidently, A then satisfies the desired conditions in Lemma 6.8.

6.4 Rewriting Theorem 5.8

We cannot immediately use Theorem 6.2 to read off the topological zeta function associated with the collection of p-adic formulae provided by Theorem 5.8 (or, more generally, [27, Thm 4.10]): the rational functions appearing in Theorem 5.8 do not, in general, give rise to elements of the algebra $\mathbf{M}$ in Theorem 6.2. In [27], we used Lemma 6.8 to rewrite the explicit formulae in [27, Thm 4.10] in a shape compatible with Theorem 6.2. In order to explicitly compute associated topological zeta functions, we now consider an algorithmic version of this rewriting process applied to the formulae in Theorem 5.8.

Let $\mathcal{T} = (\mathcal{C}_0; f_1, \ldots, f_r)$ be a non-trivial regular toric datum. Let β, V_J°, $\mathcal{C}_0^J$, and $A_J(\beta)$ be as in Theorem 5.8. Let $\mathcal{N} := \mathrm{New}(f_1 \cdots f_r)$. Since $\mathcal{T}$ is balanced, by Lemma 5.3, there exists a (unique) face $\tau \subset \mathcal{N}$ such that $\mathcal{C}_0 \subset \mathrm{N}_\tau(\mathcal{N})$; in particular, $\dim(\mathcal{C}_0) \leqslant \dim(\mathrm{N}_\tau(\mathcal{N})) = n - \dim(\tau)$. Let $\tau = \tau_1 + \cdots + \tau_r$ be the decomposition of τ into faces $\tau_i \subset \mathrm{New}(f_i)$; hence, $\tau_i = \mathrm{New}(\mathrm{in}_{\mathcal{C}_0}(f_i))$ (cf. [27, Lem. 6.1(iii)]).

For $J \subset \{1, \ldots, r\}$, let V_J be the subvariety of $\mathbf{T}_k^n$ defined by $\mathrm{in}_{\mathcal{C}_0}(f_j) = 0$ for all $j \in J$. Note that using the inclusion-exclusion principle, we may replace $\#\bar{V}_J^\circ(\mathfrak{O}_K/\mathfrak{P}_K)$ in Theorem 5.8 by $\sum_{J \subset T \subset \{1, \ldots, r\}} (-1)^{|T| + |J|} \cdot \#\bar{V}_T(\mathfrak{O}_K/\mathfrak{P}_K)$.

Again, let $J \subset \{1, \ldots, r\}$. Write $d(J) := \dim\left(\sum_{j \in J} \tau_j\right)$. Equivalently, $d(J)$ is the dimension of $\sum_{j \in J} \mathrm{New}(\mathrm{in}_{\mathcal{C}_0}(f_j)) = \mathrm{New}\left(\prod_{j \in J} \mathrm{in}_{\mathcal{C}_0}(f_j)\right)$. By Lemma 6.8, we may thus construct $B_J \in \mathrm{GL}_n(\mathbf{Z})$ and non-zero $g_j \in k[X_1^{\pm 1}, \ldots, X_{d(J)}^{\pm 1}]$ such that $g_j^{-1} \mathrm{in}_{\mathcal{C}_0}(f_j)^{B_J}$ is a Laurent monomial for each $j \in J$. Let U_J be the subvariety of $\mathbf{T}_k^{d(J)}$ defined by $g_j = 0$ for all $j \in J$ so that $V_J \approx_k U_J \times_{\mathrm{Spec}(k)} \mathbf{T}_k^{n-d(J)}$ (see Corollary 6.9). Finally, define

$$W_J(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) := \boldsymbol{q}^{-n}(\boldsymbol{q} - 1)^{n - \dim(\tau) + |J|} \cdot |\mathcal{C}_0^J|^{A_J(\beta)}(\boldsymbol{q}^{-1}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) \in \mathbf{Q}(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m).$$
$$(6.2)$$

In summary, we may rephrase Theorem 5.8 as follows.

Proposition 6.10. *Let $\mathcal{T} = (\mathcal{C}_0; f_1, \ldots, f_r)$ be regular as above. For all p-adic fields $K \supset k$ and $s_1, \ldots, s_m \in \mathbf{C}$ with $\mathrm{Re}(s_j) \geqslant 0$, unless $\mathfrak{p}_K$ belongs to some finite exceptional set (depending only on $\mathcal{T}$),*

$$\mathsf{Z}_K^{\mathcal{T}, \beta}(s_1, \ldots, s_m) = \sum_J \left(\sum_{J \subset T} (-1)^{|J| + |T|} \cdot \#\bar{U}_T(\mathfrak{O}_K/\mathfrak{P}_K) \cdot (q_K - 1)^{\dim(\tau) - d(T)} \right) \cdot W_J(q_K, q_K^{-s_1}, \ldots, q_K^{-s_m}),$$
$$(6.3)$$

where J and T range over subsets of $\{1, \ldots, r\}$. ◆

The two crucial features of Proposition 6.10 compared with Theorem 5.8 are

(i) the U_T are embedded as closed (instead of locally closed) subvarieties of tori and

(ii) each $W_J(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ belongs to the algebra $\mathbf{M}$ from §6.1 (Corollary 6.13).

6.5 Computing formal reductions modulo $q - 1$

We show that each W_J in Proposition 6.10 belongs to the $\mathbf{Q}$-algebra $\mathbf{M}$ from §6.1. While this statement alone is merely a special case of [27, Lem. 6.9(i)], the proof given

here provides an algorithm for computing $\lfloor W \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ (see Notation 6.1(ii)). For $W \in \mathbf{Q}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ and $1 \leqslant j \leqslant m$, we let $\deg_{\boldsymbol{s}_j}(W)$ denote the degree of W regarded as a univariate rational function in $\boldsymbol{s}_j$ over the field $\mathbf{Q}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{j-1}, \boldsymbol{s}_{j+1}, \ldots, \boldsymbol{s}_m)$.

Lemma 6.11. *Let $\mathcal{B}_0 \subset \mathbf{R}^r_{\geqslant 0}$ be a non-empty rational half-open cone of dimension d. Let A be an $r \times (m+1)$-matrix with entries in $\mathbf{N}_0$ and suppose that the first column of A is $(1, \ldots, 1)^\top$. Then $W(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) := (\boldsymbol{q}-1)^d |\mathcal{B}_0|^A(\boldsymbol{q}^{-1}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ belongs to the algebra $\mathbf{M}$ from §6.1. Moreover, $-d \leqslant \deg_{\boldsymbol{s}_j}(\lfloor W \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)) \leqslant 0$ for $j = 1, \ldots, m$.*

Proof. We freely use §5.4. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ consist of independent variables over $\mathbf{Q}$. We regard generating functions of rational half-open cones in $\mathbf{R}^r_{\geqslant 0}$ as elements of $\mathbf{Q}(\boldsymbol{\lambda})$.

Let $\mathcal{F}$ be a triangulation of the closure $\bar{\mathcal{B}}_0$ of $\mathcal{B}_0$ into simplicial cones. Let $\sigma \in \mathcal{F}$ and $e := \dim(\sigma)$. Since σ is simplicial, there are primitive vectors $\varrho_1, \ldots, \varrho_e \in \mathbf{N}_0^r$ with $\sigma = \mathrm{cone}(\varrho_1, \ldots, \varrho_e)$. Let $\Pi(\sigma) := \{a_1 \varrho_1 + \cdots + a_e \varrho_e : a_i \in \mathbf{R}, 0 \leqslant a_i < 1\} \cap \mathbf{Z}^r$. Then $|\sigma| = \left(\sum_{\beta \in \Pi(\sigma)} \boldsymbol{\lambda}^\beta\right) / \prod_{i=1}^e (1 - \boldsymbol{\lambda}^{\varrho_i})$. Write $\mathbf{1} := (1, \ldots, 1)$ and $A := [\mathbf{1}^\top, \alpha_1^\top, \ldots, \alpha_m^\top]$. Using the notation for monomial substitutions from §5.5, we find that

$$|\sigma|^A(\boldsymbol{q}^{-1}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) = \frac{\displaystyle\sum_{\beta \in \Pi(\sigma)} \boldsymbol{q}^{-\langle \mathbf{1}, \beta \rangle} \boldsymbol{t}_1^{\langle \alpha_1, \beta \rangle} \cdots \boldsymbol{t}_m^{\langle \alpha_m, \beta \rangle}}{\displaystyle\prod_{i=1}^e \left(1 - \boldsymbol{q}^{-\langle \mathbf{1}, \varrho_i \rangle} \boldsymbol{t}_1^{\langle \alpha_1, \varrho_i \rangle} \cdots \boldsymbol{t}_m^{\langle \alpha_m, \varrho_i \rangle}\right)}. \tag{6.4}$$

For $b \in \mathbf{N}$ and $\boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbf{N}_0^m$, let $W_{\boldsymbol{a},b}(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) := \frac{\boldsymbol{q}-1}{1 - \boldsymbol{q}^{-b} \boldsymbol{t}_1^{a_1} \cdots \boldsymbol{t}_m^{a_m}}$. Then $W_{\boldsymbol{a},b}(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) \in \mathbf{M}$ and $\lfloor W_{\boldsymbol{a},b} \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = 1/(a_1 \boldsymbol{s}_1 + \cdots + a_m \boldsymbol{s}_m + b)$ (see Definition 6.1). We conclude that $\tilde{Z}_\sigma(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) := (\boldsymbol{q}-1)^e \cdot |\sigma|^A(\boldsymbol{q}^{-1}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ belongs to $\mathbf{M}$. As $d \geqslant e$, the same is true of $Z_\sigma(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) := (\boldsymbol{q}-1)^{d-e} \cdot \tilde{Z}_\sigma(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$. Our next goal is to deduce a formula for $\lfloor Z_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ from (6.4). To that end, note that if $e < d$, then

$$\lfloor Z_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = \underbrace{\left(\lfloor (\boldsymbol{q}-1)^{d-e} \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)\right)}_{0} \cdot \left(\lfloor \tilde{Z}_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)\right) = 0.$$

If, on the other hand, $d = e$, then $Z_\sigma = \tilde{Z}_\sigma$. In order to compute $\lfloor \tilde{Z}_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$, let $f(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ be the numerator of the right-hand side of (6.4). As $f(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ is a Laurent polynomial, it belongs to $\mathbf{M}$ and satisfies $\lfloor f \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = f(1, \ldots, 1) = \#\Pi(\sigma)$; we note that $\#\Pi(\sigma)$ is the usual multiplicity of the simplicial cone σ (see [7, Prop. 11.1.8]). Using the $W_{\boldsymbol{a},b}(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ from above and (6.4), we may write

$$\tilde{Z}_\sigma(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) = f(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) \cdot \prod_{i=1}^e W_{(\langle \alpha_1, \varrho_i \rangle, \ldots, \langle \alpha_m, \varrho_i \rangle), \langle \mathbf{1}, \varrho_i \rangle}(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m). \tag{6.5}$$

Since the right-hand side of (6.5) is a product in $\mathbf{M}$, we conclude that

$$\lfloor \tilde{Z}_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = \frac{\#\Pi(\sigma)}{\prod_{i=1}^e \langle \varrho_i A, (1, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) \rangle}. \tag{6.6}$$

As we mentioned in §5.4, it is a consequence of the inclusion-exclusion principle that we may write $W(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) = \sum_{\sigma \in \mathcal{F}} c_\sigma Z_\sigma(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$, where $c_\sigma \in \mathbf{Z}$ and $c_\sigma = 1$ whenever $\dim(\sigma) = d$. In particular, since each $Z_\sigma(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ belongs to $\mathbf{M}$, $W(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m) \in \mathbf{M}$ and $\lfloor W \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = \sum_{\sigma \in \mathcal{F}} c_\sigma \lfloor Z_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$. Furthermore, as we proved above, if $\sigma \in \mathcal{F}$ satisfies $\dim(\sigma) < d$, then $\lfloor Z_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = 0$. In summary, $\lfloor W \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = \sum_\sigma \lfloor Z_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$, where the sum is taken over those $\sigma \in \mathcal{F}$ with $\dim(\sigma) = d$ only.

It only remains to establish the stated bounds for the degree of $\lfloor W \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ in $\boldsymbol{s}_j$. Let $\sigma \in \mathcal{F}$ with $\dim(\sigma) = d$. In the expression for $\lfloor \tilde{Z}_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ $(= \lfloor Z_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m))$ given in (6.6), the numerator is a positive constant and the denominator is a product of $\dim(\sigma)$ factors of the form $a_1 \boldsymbol{s}_1 + \cdots + a_m \boldsymbol{s}_m + b$ for $a_1, \ldots, a_m \in \mathbf{N}_0$ and $b \in \mathbf{N}$. By the non-negativity of all these numbers, the degree of $\lfloor W \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ in $\boldsymbol{s}_j$ is simply the maximal degree of any $\lfloor Z_\sigma \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ in $\boldsymbol{s}_j$ for $\sigma \in \mathcal{F}$ with $\dim(\sigma) = d$. ♦

Remark 6.12.

(i) In [11, §5], Denef and Loeser gave an explicit convex-geometric formula for the topological zeta function associated with a suitably non-degenerate polynomial. In view of the p-adic formulae of Denef and Hoornaert [10], the explicit descriptions of the rational functions $J(\tau, s) \in \mathbf{Q}(s)$ in terms of triangulations in [11] can be regarded as a special case of Lemma 6.11.

(ii) The proof of Lemma 6.11 shows that the rational function $\lfloor W \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ only depends on the closure of $\mathcal{B}_0$.

The following corollary will allow us to derive a topological counterpart of Proposition 6.10, namely Proposition 6.14.

Corollary 6.13. *In the setting of §6.4, for each $J \subset \{1, \ldots, r\}$, the rational function $W_J(\boldsymbol{q}, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$ belongs to $\mathbf{M}$. Moreover, $\lfloor W_J \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = 0$ if and only if $\dim(\mathcal{C}_0^J) < n - \dim(\tau) + |J|$.*

Proof. As $\dim(\mathcal{C}_0) \leqslant n - \dim(\tau)$ and $\mathcal{C}_0^J \subset \mathcal{C}_0 \times \mathbf{R}_{>0}^{|J|}$, we conclude that $\dim(\mathcal{C}_0^J) \leqslant n - \dim(\tau) + |J|$. ♦

6.6 Computing Euler characteristics

Let $f_1, \ldots, f_r \in k[\boldsymbol{X}^{\pm 1}]$ and let U be the closed subvariety of $\mathbf{T}_k^n$ defined by $f_1 = \cdots = f_r = 0$. The typical example to bear in mind is the case where $f_1, \ldots, f_r$ are the initial forms of a non-empty regular toric datum. We now consider the computation of the topological Euler characteristic $\chi(U(\mathbf{C}))$. The function EVALUATETOPOLOGICALLY (see §6.7) will rely on our ability to compute these numbers.

General methods. Aluffi [2] described an algorithm for computing the topological Euler characteristic of a not necessarily smooth projective variety in characteristic zero based on the computation of so-called Chern-Schwartz-MacPherson classes; for recent developments,

see [18, 21]. In principle, such general algorithms can be used to compute $\chi(U(\mathbf{C}))$ from above. Indeed, after clearing denominators, we may assume that $f_1, \ldots, f_r \in k[\boldsymbol{X}]$. Let $\tilde{f}_i \in k[X_0, \ldots, X_n]$ denote the homogenisation of f_i. Let the subvarieties V and W of n-dimensional projective space over k be defined by $\tilde{f}_1 = \cdots = \tilde{f}_r = 0$ and $\tilde{f}_1 = \cdots = \tilde{f}_r = X_0 \cdots X_n = 0$, respectively. Then $U \approx_k V \setminus W$ and so $\chi(U(\mathbf{C})) = \chi(V(\mathbf{C})) - \chi(W(\mathbf{C}))$; cf. [2, §2.8].

In practice, while implementations of [2, 18, 21] exist, these methods are usually too costly for our applications to the computation of topological zeta functions. For example, the computation of $\zeta_{\mathrm{Fil}_4,\mathrm{top}}(\boldsymbol{s})$ previously announced in [27, §7.3] involves the Euler characteristics of thousands of subvarieties of $\mathbf{T}_{\mathbf{Q}}^{15}$. In our implementation (see §8), we therefore attempt to compute Euler characteristics using special-purpose methods.

The Bernstein-Khovanskii-Kushnirenko Theorem. As we already exploited in [27, §6], if $(f_1, \ldots, f_r)$ is non-degenerate in the sense of Khovanskii [22, §2], then [22, §3, Thm 2] provides an explicit formula for $\chi(U(\mathbf{C}))$ in terms of various mixed volumes associated with the Newton polytopes of $f_1, \ldots, f_r$; we refer to [6, Ch. 7, §§4–5] for details on mixed volumes (including explicit examples). Since U, and hence $\chi(U(\mathbf{C}))$, only depends on the radical of the ideal generated by $f_1, \ldots, f_r$ within $k[\boldsymbol{X}^{\pm 1}]$, when $(f_1, \ldots, f_r)$ is degenerate in Khovanskii's sense, we can try to use standard techniques such as multivariate polynomial division (after clearing denominators) and saturation to simplify $(f_1, \ldots, f_r)$, e.g. by reducing $\sum_{i=1}^r |\mathrm{supp}(f_i)|$.

Decomposing subvarieties of tori. If $(f_1, \ldots, f_r)$ remains degenerate after applying the simplification steps indicated above, we try to decompose U as follows. Suppose that after renumbering $f_1, \ldots, f_r$ (or, more generally, a suitable application of a matrix from $\mathrm{GL}_n(\mathbf{Z})$ as in §6.3) and rescaling of f_1 by Laurent monomials (which does not change U), we have $f_1 = X_n - w$ for $w \in k[X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}]$. Let $V \subset \mathbf{T}_k^{n-1}$ be the subvariety defined by $f_2(X_1, \ldots, X_{n-1}, w) = \cdots = f_r(X_1, \ldots, X_{n-1}, w) = 0$ and let $W \subset V$ be defined by $w = 0$. Then $U \approx_k V \setminus W$ and we can recursively try to compute $\chi(U(\mathbf{C})) = \chi(V(\mathbf{C})) - \chi(W(\mathbf{C}))$ using the techniques mentioned above.

While there is, in general, little reason to expect the specialised methods from the preceding two paragraphs to succeed, in practice, the author found that combining these simple methods often suffices to compute Euler characteristics in TopologicalZeta-Function.

6.7 An algorithm for computing topological zeta functions associated with regular toric data

The following is a topological version of Proposition 6.10.

Proposition 6.14. *Let the notation be as in Proposition 6.10; in particular,* $\mathcal{T} = (\mathcal{C}_0; f_1, \ldots, f_r)$ *is a regular toric datum. Then*

$$\mathsf{Z}_{\mathrm{top}}^{\mathcal{T},\beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = \sum_{\substack{J \subset T \subset \{1,\ldots,r\}, \\ n - d(T) + |J| = \dim(\mathcal{C}_0^J)}} (-1)^{|J|+|T|} \cdot \chi(U_T(\mathbf{C})) \cdot \lfloor W_J \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m).$$

Proof. Let $J \subset T$. Proposition 6.10 and Corollary 6.13 show that $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T},\beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ is the sum of $(-1)^{|J|+|T|} \cdot \chi(U_T(\mathbf{C})) \cdot \lfloor W_J \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ over pairs $J \subset T$ with $d(T) = \dim(\tau)$ and $n - \dim(\tau) + |J| = \dim(\mathcal{C}_0^J)$. The latter two conditions are both satisfied if and only if $n - d(T) + |J| = \dim(\mathcal{C}_0^J)$ since $d(T) \leqslant \dim(\tau)$ and $n - \dim(\tau) + |J| \geqslant \dim(\mathcal{C}_0^J)$. ♦

The Euler characteristics $\chi(U_T(\mathbf{C}))$ can be determined as in §6.6, while the rational functions $\lfloor W_J \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ may be computed as explained in §6.5. We obtain the following algorithm.

The function EVALUATETOPOLOGICALLY. We let EVALUATETOPOLOGICALLY denote the function which, given a regular toric datum $\mathcal{T}$ in n variables over k and a matrix $\beta \in \mathrm{M}_{m \times n}(\mathbf{N}_0)$, computes $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T},\beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ using Proposition 6.14. Specifically, for each $J \subset \{1, \ldots, r\}$, we use §6.3 and §6.6 to first compute

$$e_J := \sum_{\substack{J \subset T \subset \{1,\ldots,r\}, \\ n - d(T) + |J| = \dim(\mathcal{C}_0^J)}} (-1)^{|J|+|T|} \cdot \chi(U_T(\mathbf{C})).$$

Only if e_J turns out to be non-zero, do we proceed to compute $\lfloor W_J \rfloor(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$ using a triangulation of the closure of $\mathcal{C}_0^J$ as in the proof of Lemma 6.11.

Remark 6.15.

(i) The topological zeta functions that we seek to compute can be written as univariate specialisations of topological zeta functions associated with toric data; see Remark 3.4. In practice, we avoid the costly multivariate rational function arithmetic altogether and apply these specialisations directly in the triangulation step of EVALUATETOPOLOGICALLY; for a theoretical justification, use [27, Rem. 5.15].

(ii) If we ignore the simplification step in line 4 of TOPOLOGICALZETAFUNCTION (Algorithm 4.1), then, at this point, we have obtained an algorithmic version of [27, Thm 6.7] (restricted to the integrals considered here). Namely, let $(\mathcal{C}_0; \boldsymbol{f})$ be a toric datum as in Algorithm 4.1 and suppose that $\boldsymbol{f}$ is globally non-degenerate in the sense of [27, Def. 4.2(ii)]. It follows from [27, Lem. 6.1] that each $\mathcal{T} \in \mathrm{BALANCE}(\mathcal{C}_0; \boldsymbol{f})$ is regular and that the defining polynomials of the varieties U_T in EVALUATETOPOLOGICALLY$(\mathcal{T}, \beta)$ satisfy Khovanskii's non-degeneracy conditions; the computation of e_J as part of EVALUATETOPOLOGICALLY is then a direct implementation of [27, Prop. 6.5].

7 Simplification and reduction

We now describe the remaining two functions SIMPLIFY and REDUCE invoked by TOPO-LOGICALZETAFUNCTION (Algorithm 4.1).

7.1 Weak and strong equivalence of toric data

Definition 7.1. Let $(\mathcal{C}_0; \boldsymbol{f})$ and $(\mathcal{D}_0; \boldsymbol{g})$ be toric data over k.

(i) $(\mathcal{C}_0; \boldsymbol{f})$ and $(\mathcal{D}_0; \boldsymbol{g})$ are **strongly equivalent** if $\mathcal{C}_0 = \mathcal{D}_0$ and there is a finite $S \subset \operatorname{Spec}(\mathfrak{o})$ such that if $K \supset k$ is a p-adic field with $\mathfrak{p}_K \notin S$, then $\|\boldsymbol{f}(\boldsymbol{x})\|_K = \|\boldsymbol{g}(\boldsymbol{x})\|_K$ for all $\boldsymbol{x} \in \mathbf{T}^n(K)$ with $\nu_K(\boldsymbol{x}) \in \mathcal{C}_0$.

(ii) $(\mathcal{C}_0; \boldsymbol{f})$ and $(\mathcal{D}_0; \boldsymbol{g})$ are **weakly equivalent** if there is a finite $S \subset \operatorname{Spec}(\mathfrak{o})$ such that $(\mathcal{C}_0; \boldsymbol{f})_K = (\mathcal{D}_0; \boldsymbol{g})_K$ for all p-adic fields $K \supset k$ with $\mathfrak{p}_K \notin S$.

We let $\sim_s$ and $\sim_w$ signify strong and weak equivalence, respectively.

Strong equivalence implies weak one but the converse is false; for example, $\left(\mathbf{R}_{\geqslant 0}; X_1^{-1}\right)$ and $\left(\mathbf{R}_{\geqslant 0}; X_1^{-1} - X_1\right)$ are weakly equivalent but not strongly so. Theorem 6.2 yields the following.

Lemma 7.2. *Let $\mathcal{T}$ and $\mathcal{T}'$ be weakly equivalent toric data in n variables over K and let $\beta \in \mathrm{M}_{m \times n}(\mathbf{N}_0)$. Then $\mathsf{Z}_{\mathrm{top}}^{\mathcal{T}, \beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) = \mathsf{Z}_{\mathrm{top}}^{\mathcal{T}', \beta}(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$.* ♦

We now collect some instances of these equivalences in a form that resembles Gaussian elimination and the multivariate polynomial division algorithm (see e.g. [1, §1.5]). By a **term**, we mean a Laurent polynomial of the form $c \boldsymbol{X}^\alpha$, where $c \in k^\times$ and $\alpha \in \mathbf{Z}^n$. Given a rational half-open cone $\mathcal{C}_0 \subset \mathbf{R}_{\geqslant 0}^n$, its **dual** $\mathcal{C}_0^* := \{\omega \in \mathbf{R}^n : \langle \alpha, \omega \rangle \geqslant 0 \text{ for all } \alpha \in \mathcal{C}_0\}$ is a rational *closed* cone which contains $\mathbf{R}_{\geqslant 0}^n$. As usual, we let $\mathcal{C}_0^\perp := \{\omega \in \mathbf{R}^n : \langle \alpha, \omega \rangle = 0 \text{ for all } \alpha \in \mathcal{C}_0\}$. The unit group $k[\mathcal{C}_0^* \cap \mathbf{Z}^n]^\times$ of $k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ (see §5.4) consists precisely of those terms $c \boldsymbol{X}^\alpha$ with $c \in k^\times$ and $\alpha \in \mathcal{C}_0^\perp \cap \mathbf{Z}^n$. In the following, we assume that $r \in \mathbf{N}_0$ is large enough for the statements given to make sense.

Lemma 7.3. *Let $(\mathcal{C}_0; \boldsymbol{f}) = (\mathcal{C}_0; f_1, \ldots, f_r)$ be a toric datum over k. Then:*

(S0) $(\mathcal{C}_0; f_1, \ldots, f_r, 0) \sim_s (\mathcal{C}_0; f_1, \ldots, f_r)$.

(S1) $(\mathcal{C}_0; f_1, \ldots, f_r) \sim_s (\mathcal{C}_0; f_{1\sigma}, \ldots, f_{r\sigma})$ *for every* $\sigma \in \operatorname{Sym}(r)$.

(S2) *If* $u \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]^\times$ *and* $v \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$, *then* $(\mathcal{C}_0; \boldsymbol{f}) \sim_s (\mathcal{C}_0; u f_1 + v f_2, f_2, \ldots, f_r)$.

(S3) *If* f_1 *is* $\mathcal{C}_0$-*balanced and* $\operatorname{in}_{\mathcal{C}_0}(f_1)$ *is a term, then* $(\mathcal{C}_0; \boldsymbol{f}) \sim_s (\mathcal{C}_0; \operatorname{in}_{\mathcal{C}_0}(f_1), f_2, \ldots, f_r)$.

(W1) *If* $v \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$, *then* $(\mathcal{C}_0; \boldsymbol{f}) \sim_w (\mathcal{C}_0; f_1 + v, f_2, \ldots, f_r)$.

(W2) *If* $f_1 = c \boldsymbol{X}^\alpha$ *for* $c \in k^\times$ *and* $\alpha \in \mathbf{Z}^n$, *then* $(\mathcal{C}_0; \boldsymbol{f}) \sim_w (\mathcal{C}_0 \cap \{\alpha\}^*; f_2, \ldots, f_r)$.

Proof. (S0) and (S1) are obvious. We precede our proofs of the other cases with some general remarks.

Let $K \supset k$ be a p-adic field. Let $\omega \in \mathbf{Z}^n$ and $\boldsymbol{x} \in \mathbf{T}^n(K)$ with $\nu_K(\boldsymbol{x}) = \omega$. Recall that $\pi_K \in \mathfrak{P}_K \setminus \mathfrak{P}_K^2$ denotes a fixed uniformiser. Write $\boldsymbol{x} = (\pi_K^{\omega_1} u_1, \ldots, \pi_K^{\omega_n} u_n)$ for $\boldsymbol{u} \in \mathbf{T}^n(\mathfrak{O}_K)$. For every non-zero $g \in \mathfrak{O}_K[\boldsymbol{X}^{\pm 1}]$ and arbitrary $\alpha \in \operatorname{supp}(\operatorname{in}_\omega(g))$, there exists $e \in \mathfrak{P}_K$ with $g(\boldsymbol{x}) = \pi_K^{\langle \alpha, \omega \rangle} \cdot (\operatorname{in}_\omega(g)(\boldsymbol{u}) + e)$; cf. [27, §4.1]. Hence, if $u \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]^\times$, $v \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$, and $\boldsymbol{x} \in \mathbf{T}^n(K)$ with $\nu_K(\boldsymbol{x}) \in \mathcal{C}_0$, then $|u(\boldsymbol{x})|_K = 1$ and $|v(\boldsymbol{x})|_K \leqslant 1$, provided that the unique non-zero coefficient of u is a $\mathfrak{P}_K$-adic unit and all coefficients of v are $\mathfrak{P}_K$-adic integers.

If $a \in K$ and $e \in \mathfrak{O}_K$, then $|a + e|_K \leqslant 1$ if and only if $|a|_K \leqslant 1$ whence (W1) follows. Also, (S2) follows since if $a, b \in K$ and $e \in \mathfrak{O}_K$, then $\|a, b\|_K = \|a + eb, b\|_K$. Indeed, if $|a|_K \leqslant |eb|_K$, then both $|a|_K, |a + eb|_K \leqslant |b|_K$; if, on the other hand, $|a|_K > |eb|_K$, then $|a + eb|_K = |a|_K$.

Finally, for (S3) and (W2), let f_1 be $\mathcal{C}_0$-balanced with $\operatorname{in}_{\mathcal{C}_0}(f_1) = c\boldsymbol{X}^\alpha$, where $c \in k^\times$ and $\alpha \in \mathbf{Z}^n$; we may assume that $c \in \mathfrak{O}_K^\times$ and $f_1 \in \mathfrak{O}_K[\boldsymbol{X}^{\pm 1}]$. If $x \in \mathbf{T}^n(K)$ with $\nu_K(\boldsymbol{x}) = \omega \in \mathcal{C}_0 \cap \mathbf{Z}^n$, then $|f_1(\boldsymbol{x})|_K = q_K^{-\langle \alpha, \omega \rangle} = |c\boldsymbol{x}^\alpha|_K$. Hence, $|f_1(\boldsymbol{x})|_K \leqslant 1$ if and only if $\langle \alpha, \omega \rangle \geqslant 0$. $\blacklozenge$

7.2 Simplification

What we call simplification is the systematic application of Lemma 7.3 (with the exception of operation (S2); see Proposition 7.5) to toric data.

Definition 7.4. A toric datum $(\mathcal{C}_0; f_1, \ldots, f_r)$ is **simple** if the following conditions are satisfied for $i = 1, \ldots, r$:

   (i) $f_i \neq 0$ and no term of f_i lies in $k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$.

   (ii) If $f_i f_j^{-1} \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ for $1 \leqslant j \leqslant r$, then $i = j$.

   (iii) If f_i is $\mathcal{C}_0$-balanced, then $\operatorname{in}_{\mathcal{C}_0}(f_i)$ consists of at least two terms.

The function SIMPLIFY. We now describe the function SIMPLIFY used in TOPO-LOGICALZETAFUNCTION (Algorithm 4.1). Given a toric datum $(\mathcal{C}_0; f_1, \ldots, f_r)$, we remove those terms of each f_i that lie in $k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$. We then discard those f_j with $f_j = 0$ altogether. Next, if $f_i f_j^{-1} \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ for $i \neq j$, then we discard f_i. Finally, if some f_i is $\mathcal{C}_0$-balanced with $\operatorname{in}_{\mathcal{C}_0}(f_i) = c\boldsymbol{X}^\alpha$ (where $c \in k^\times$), then we discard f_i and replace $\mathcal{C}_0$ by $\mathcal{C}_0 \cap \{\alpha\}^*$. Since shrinking $\mathcal{C}_0$ enlarges its dual $\mathcal{C}_0^*$, further terms might now become redundant. We therefore repeatedly apply the above process until $(\mathcal{C}_0; f_1, \ldots, f_r)$ stabilises. As each non-trivial operation decreases $r + \sum_{i=1}^r \#\operatorname{supp}(f_i)$, after finitely many steps, we obtain a simple toric datum which is weakly equivalent to the original $(\mathcal{C}_0; f_1, \ldots, f_r)$ by Lemma 7.3.

Note that Lemma 7.3(S2) is only used in a very simple form. Namely, discarding f_i when $f_i f_j^{-1} \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ for $i \neq j$ constitutes an application of (S0)–(S2). The reason we only made very limited use of (S2) is to ensure the following:

Proposition 7.5. *Let $\mathcal{T}$ be a toric datum.*

(i) *If $\mathcal{T}$ is balanced, then so is* SIMPLIFY($\mathcal{T}$).

(ii) *If $\mathcal{T}$ is regular, then so is* SIMPLIFY($\mathcal{T}$).

Proof. Let $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$ be non-trivial. The properties of being balanced or regular are preserved if we discard polynomials, shrink $\mathcal{C}_0$, or remove non-initial terms. Let $f \in \boldsymbol{f}$ be $\mathcal{C}_0$-balanced and suppose that $\mathrm{in}_{\mathcal{C}_0}(f)$ contains a term $c\boldsymbol{X}^\alpha$ with $c \in k^\times$ and $\alpha \in \mathcal{C}_0^*$. Then $0 \leqslant \langle \alpha, \omega \rangle \leqslant \langle \beta, \omega \rangle$ for all $\beta \in \mathrm{supp}(f)$ and $\omega \in \mathcal{C}_0$ whence SIMPLIFY($\mathcal{T}$) will discard f entirely. ♦

7.3 Reduction

We now describe the reduction step in TOPOLOGICALZETAFUNCTION (Algorithm 4.1). The function REDUCE takes as input a balanced (Definition 5.1(ii)) and simple (Definition 7.4) toric datum $\mathcal{T} = (\mathcal{C}_0; f_1, \ldots, f_r)$ which is singular (Definition 5.5). We therefore cannot directly use EVALUATETOPOLOGICALLY from §6.7 to compute the associated topological zeta function. Our goal is to construct and return a partition (Definition 5.4) $\mathfrak{C}$ of $\mathcal{T}$. Ideally, we would like $\mathfrak{C}$ to consist of regular toric data but our immediate goal is more modest: we systematically construct *some* non-trivial partition $\mathfrak{C}$ in the hope that repeated further applications of BALANCE, SIMPLIFY, and REDUCE to its members in the main loop of TOPOLOGICALZETAFUNCTION will eventually produce regular toric data only. Success of this procedure is not guaranteed and we need to allow REDUCE to fail (at which point TOPOLOGICALZETAFUNCTION will fail too) in order to guarantee termination.

Reduction candidates. We begin by isolating a source of the singularity of $\mathcal{T}$. Namely, the method for regularity testing in §5.3 readily provides us with an inclusion-minimal set $J \subset \{1, \ldots, r\}$ such that the Jacobian matrix of $(\mathrm{in}_{\mathcal{C}_0}(f_j))_{j \in J}$ has rank less than $|J|$ at some point $\boldsymbol{u} \in \mathbf{T}^n(\bar{k})$ with $\mathrm{in}_{\mathcal{C}_0}(f_j)(\boldsymbol{u}) = 0$ for all $j \in J$. After renumbering $f_1, \ldots, f_r$, we may assume that $J = \{1, \ldots, e\}$. If $e = 1$, then we give up and let REDUCE fail. Suppose that $e \geqslant 2$. By a **reduction candidate** for $\mathcal{T}$ we mean a quadruple (i, j, t_i, t_j), where $1 \leqslant i < j \leqslant e$, t_i is a term of $\mathrm{in}_{\mathcal{C}_0}(f_i)$, and t_j is a term of $\mathrm{in}_{\mathcal{C}_0}(f_j)$.

We note that the content of this section arose from the author's experimental observation that in various cases of interest (see, for example, the computation of $\zeta_{\mathrm{Fil}_4,\mathrm{top}}(\boldsymbol{s})$ in §9.1), there exists a reduction candidate (i, j, t_i, t_j) such that $\mathrm{in}_{\mathcal{C}_0}(f_i)$ and $\mathrm{in}_{\mathcal{C}_0}(f_j)$ coincide up to multiplication by a unit of $k[\boldsymbol{X}^{\pm 1}]$.

Performing reduction. Let (i, j, t_i, t_j) be a reduction candidate for $\mathcal{T}$. Let α_i and α_j denote the exponent vectors of the monomial in t_i and t_j, respectively. We decompose $\mathcal{C}_0$

into two pieces $\mathcal{C}_0^{\leqslant} := \mathcal{C}_0 \cap \{\alpha_j - \alpha_i\}^*$ and $\mathcal{C}_0^{>} := \mathcal{C}_0 \setminus \mathcal{C}_0^{\leqslant}$, both being rational half-open cones. Since $\mathcal{T}$ is balanced, the restriction of the linear form $\langle \alpha_i, - \rangle$ to $\mathcal{C}_0$ only depends on f_i and not on the chosen term t_i, and similarly for α_j. In particular, the decomposition $\mathcal{C}_0 = \mathcal{C}_0^{\leqslant} \cup \mathcal{C}_0^{>}$ only depends on (i, j). Define toric data

$$\mathcal{T}^{\leqslant} := \Big(\mathcal{C}_0^{\leqslant}; \; f_1, \ldots, f_{j-1}, \; f_j - \frac{t_j}{t_i} f_i, \; f_{j+1}, \ldots, f_r \Big),$$
$$\mathcal{T}^{>} := \Big(\mathcal{C}_0^{>}; \; f_1, \ldots, f_{i-1}, \; f_i - \frac{t_i}{t_j} f_j, \; f_{i+1}, \ldots, f_r \Big).$$

By construction, $t_j/t_i \in k\big[(\mathcal{C}_0^{\leqslant})^* \cap \mathbf{Z}^n\big]$ and $t_i/t_j \in k\big[(\mathcal{C}_0^{>})^* \cap \mathbf{Z}^n\big]$ so Lemma 7.3(S2) shows that $\{\mathcal{T}^{\leqslant}, \mathcal{T}^{>}\}$ is a partition of $\mathcal{T}$. Having chosen a reduction candidate (i, j, t_i, t_j), we let REDUCE$(\mathcal{T})$ return $\{\mathcal{T}^{\leqslant}, \mathcal{T}^{>}\}$.

The name "reduction" given to the procedure described here is due to the similarity to reduction steps in the theory of Gröbner bases; see e.g. [1, §1.5]. There are, however, substantial differences between the two procedures. Most importantly, the role of divisibility relations "$t_i \mid t_j$" between terms in polynomial algebras in the classical setting is replaced by an integrality condition "$t_j/t_i \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$" for Laurent terms. In the present setting, we can enforce arbitrary divisibility relations of this form by cutting $\mathcal{C}_0$ in half—at the cost of having to consider the opposite relation as well.

Finding reduction candidates. It remains to explain a strategy for choosing a reduction candidate (i, j, t_i, t_j) for $\mathcal{T}$. This is the most critical part of the entire reduction step and it may well fail. Based on experimental experience, we use a greedy approach which seeks to minimise the following invariant. Define the **weight** of a balanced toric datum $\mathcal{T}' = (\mathcal{D}_0; g_1, \ldots, g_u)$ to be $\mathrm{w}(\mathcal{T}') := \sum_{d=1}^u \#\mathrm{supp}\big(\mathrm{in}_{\mathcal{D}_0}(g_d)\big)$. For each reduction candidate (i, j, t_i, t_j), we construct the associated partition $\{\mathcal{T}^{\leqslant}, \mathcal{T}^{>}\}$ of $\mathcal{T}$ as indicated above. Using BALANCE and SIMPLIFY, we then further refine this partition to produce a partition, $\mathfrak{C}(i, j, t_i, t_j)$ say, of $\mathcal{T}$ which consists of balanced and simple toric data. Let $\mathfrak{C}'(i, j, t_i, t_j) \subset \mathfrak{C}(i, j, t_i, t_j)$ be the subset of singular toric data. If $\mathfrak{C}'(i, j, t_i, t_j) = \varnothing$ for some (i, j, t_i, t_j), then we use such a quadruple as our reduction candidate. Otherwise, we choose (i, j, t_i, t_j) such that the average weight

$$\Big(\sum_{\mathcal{T}' \in \mathfrak{C}'(i,j,t_i,t_j)} \mathrm{w}(\mathcal{T}') \Big) / |\mathfrak{C}'(i, j, t_i, t_j)|$$

of singular toric data is minimal. In practice, we then of course let REDUCE$(\mathcal{T})$ return $\mathfrak{C}(i, j, t_i, t_j)$ instead of $\{\mathcal{T}^{\leqslant}, \mathcal{T}^{>}\}$. While alternative choices exist, the author found this specific strategy for choosing reduction candidates to be particularly useful in practice.

In order to ensure termination of TOPOLOGICALZETAFUNCTION, we assign a **depth** to each toric datum. The initial toric datum given as the input of TOPOLOGICALZETA-FUNCTION has depth 0. We further let BALANCE$(\mathcal{T})$ and SIMPLIFY$(\mathcal{T})$ return toric data of the same depth as $\mathcal{T}$. If, having chosen (i, j, t_i, t_j) as part of the reduction step, $\mathrm{w}(\mathcal{T}') > \mathrm{w}(\mathcal{T})$ for some $\mathcal{T}' \in \mathfrak{C}'(i, j, t_i, t_j)$, then we increase the depth of $\mathcal{T}'$. Termination is guaranteed by letting REDUCE$(\mathcal{T})$ fail whenever the depth of $\mathcal{T}$ exceeds some constant

value. While this approach is less elegant than a strictly greedy approach, where we would, for example, insist that $w(\mathcal{T}') < w(\mathcal{T})$ for all $\mathcal{T}' \in \mathfrak{C}'(i, j, t_i, t_j)$, it is more powerful in practice. Indeed, our depth-based approach might incorporate several intermediate reduction steps which do not provide any immediate improvements (in the sense that the weight is not lowered). However, by carrying out such seemingly useless intermediate steps, new reduction candidates might then arise and these might subsequently lead to a lower weight.

For an illustration of the reduction step applied to a "real-life" example, see §9.1.

8 Practical matters

8.1 Introducing "Zeta"

The Python-package Zeta [28] for Sage [31] provides an implementation of Topologi-calZetaFunction (Algorithm 4.1) for computing topological subalgebra, ideal, and submodule zeta functions for $k = \mathbf{Q}$. Sage natively supports computations with rational polyhedra and we use these capabilities to simulate computations with half-open cones (see §8.4). Polynomial arithmetic and Gröbner bases computations are handled by Singular [16]. For the computations of mixed volumes mentioned in §6.6, we use Gfan [20]. While Sage provides functionality for computing triangulations, we use the fast implementation provided by Normaliz [5] if it is available. In order to use Zeta to compute topological zeta functions associated with nilpotent groups via Theorem 2.3, one may use the GAP-package Guarana [3, 32] which provides an effective version of the Mal'cev correspondence.

8.2 On the scope of Algorithm 4.1 and its implementation

Theoretical limitations. While TopologicalZetaFunction allows us to compute many more topological zeta functions than [27, Thm 6.7] alone could, it is fairly easy to produce examples that seem completely resistant to our approach. For example, while the vast majority of known topological and local subalgebra and ideal zeta functions arise from nilpotent Lie rings, to the author's knowledge, not a single example of any such zeta function associated with a nilpotent Lie ring of class $\geqslant 5$ has ever been computed. In particular, there are various examples of nilpotent Lie rings of additive rank 6 whose topological and local subring and ideal zeta functions remain unknown—our method has so far been unable to remedy this. As the additive rank of the possibly non-associative ring under consideration increases or the assumption that it be nilpotent and Lie is relaxed, examples amenable to our method become rare.

Practical issues. So far, the most successful applications of Zeta were concerned with (nilpotent) associative, commutative, or Lie rings of additive rank at most 6. Even in the case of nilpotent Lie rings of rank 6, some of the computations carried out by the author took several months to complete (using 16 parallel processes on the

computer indicated at the end of §9.1 (see §8.3)). In such cases, the most expensive step of TOPOLOGICALZETAFUNCTION is the final line. At this point, regular will be populated with possibly thousands of regular toric data. For each $(\mathcal{D}_0; g_1, \ldots, g_e) \in$ regular, we then consider each of the half-open cones $\mathcal{D}_0^J$ indexed by $J \subset \{1, \ldots, e\}$ yielding perhaps tens of thousands of half-open cones in total. Finally, the triangulation step in EVALUATETOPOLOGICALLY will often decompose each $\bar{\mathcal{D}}_0^J$ into possibly tens or even hundreds of thousands of simplicial cones; note that for examples of rank 6, the ambient Euclidean space of each $\mathcal{D}_0^J$ will have dimension at least $21 = \frac{6 \cdot 7}{2}$.

8.3 EVALUATETOPOLOGICALLY in practice

As mentioned in Remark 6.15(i), in our implementation of EVALUATETOPOLOGICALLY we immediately apply specialisations of the form $s_j \mapsto s - j$ (see Remark 3.4) needed to recover the desired univariate topological zeta function. Moreover, in order to avoid costly rational function arithmetic, we do not actually carry out either the summation in line 14 of Algorithm 14 nor that in the proof of Lemma 6.11. Instead, we first compute the final output of TOPOLOGICALZETAFUNCTION, $Z_{\mathrm{top}}(s) \in \mathbf{Q}(s)$ say, as an unevaluated (possibly large) sum of rational functions of the form $\frac{c}{(a_1 s - b_1) \cdots (a_d s - b_d)}$ for suitable integers a_i, b_i, c (arising from simplicial cones in Lemma 6.11 and Euler characteristics in §6.6). As we construct these rational functions, we keep track of a candidate denominator of $Z_{\mathrm{top}}(s)$, i.e. a polynomial $g \in \mathbf{Z}[s]$, $g \neq 0$ with $g Z_{\mathrm{top}}(s) \in \mathbf{Z}[s]$. Using Lemma 6.5, we may then recover $Z_{\mathrm{top}}(s)$ using random evaluation and polynomial interpolation.

After successful termination of the main loop (lines 1–13) in Algorithm 4.1, the remaining tasks of computing Euler characteristics, triangulating cones, and evaluating rational functions can be trivially parallelised; our implementation makes use of this.

8.4 Computing with half-open cones

We defined our basic data structure, the toric data from §3, in terms of rational half-open cones $\mathcal{C}_0 \subset \mathbf{R}_{\geqslant 0}^n$ since they constitute the smallest collection of subsets of $\mathbf{R}^n$ which contains $\mathbf{R}_{\geqslant 0}^n$ and which is stable under the effects of BALANCE, SIMPLIFY, and REDUCE. However, half-open cones and polyhedra (rational or not) are scarcely used in the literature and they are usually not directly supported by existing software. Apart from triangulating closed rational cones, the only computational tasks involving half-open cones that we actually relied upon are the following:

(i) Compute the intersection of two rational half-open cones.

(ii) Decide if a rational half-open cone is empty.

(iii) Construct the closure of a non-empty rational half-open cone.

(iv) Decide membership of a vector in the dual of a rational half-open cone.

As $\mathcal{C}_0^* = \bar{\mathcal{C}}_0^*$ for a half-open cone $\mathcal{C}_0$, (iii) reduces (iv) to the closed case (which is standard).

Definition 8.1. By a (polyhedral) **model** of a rational half-open cone $\mathcal{C}_0 \subset \mathbf{R}^n$, we mean a rational polyhedron $\mathcal{P}_0 \subset \mathbf{R}^n$ such that $\mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{P}_0 \cap \mathbf{Z}^n$ and $a\boldsymbol{x} \in \mathcal{P}_0$ for all $a \in \mathbf{N}$ and $\boldsymbol{x} \in \mathcal{P}_0$.

For example, for each $b \in \mathbf{N}$, the closed interval $[1/b, \infty)$ is a model of the open interval $(0, \infty)$. As we will now explain, we may replace half-open cones by models in our computations.

Proposition 8.2. *Let $\mathcal{C}_0, \mathcal{C}_0' \subset \mathbf{R}^n$ be rational half-open cones.*

(i) *There exists a model of $\mathcal{C}_0$.*

(ii) *Let $\mathcal{P}_0$ be a model of $\mathcal{C}_0$ and $\mathcal{P}_0'$ be a model of $\mathcal{C}_0'$. Then $\mathcal{P}_0 \cap \mathcal{P}_0'$ is a model of $\mathcal{C}_0 \cap \mathcal{C}_0'$.*

(iii) *Let $\mathcal{P}_0$ be a model of $\mathcal{C}_0$. Then $\mathcal{C}_0$ is empty if and only if $\mathcal{P}_0$ is empty.*

(iv) *If $\mathcal{C}_0 \neq \varnothing$ and $\mathcal{P}_0$ is a model of $\mathcal{C}_0$, then $\bar{\mathcal{C}}_0$ is the smallest cone containing $\mathcal{P}_0$.*

(v) *If $\mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{C}_0' \cap \mathbf{Z}^n$, then $\mathcal{C}_0 = \mathcal{C}_0'$. Hence, $\mathcal{C}_0$ is determined by any of its models.*

Proof.

(i) There are finitely many $\phi_i, \chi_j \in \mathbf{Z}^n$ ($i \in I, j \in J$) with

$$\mathcal{C}_0 = \{\omega \in \mathbf{R}^n : \forall i \in I, \langle \phi_i, \omega \rangle \geqslant 0 \text{ and } \forall j \in J, \langle \chi_j, \omega \rangle > 0\}$$

and $\{\omega \in \mathbf{R}^n : \forall i \in I, \langle \phi_i, \omega \rangle \geqslant 0 \text{ and } \forall j \in J, \langle \chi_j, \omega \rangle \geqslant 1\}$ is a model of $\mathcal{C}_0$.

(ii) Observe $(\mathcal{C}_0 \cap \mathcal{C}_0') \cap \mathbf{Z}^n = (\mathcal{C}_0 \cap \mathbf{Z}^n) \cap (\mathcal{C}_0' \cap \mathbf{Z}^n) = (\mathcal{P}_0 \cap \mathbf{Z}^n) \cap (\mathcal{P}_0' \cap \mathbf{Z}^n) = (\mathcal{P}_0 \cap \mathcal{P}_0') \cap \mathbf{Z}^n$. Moreover, as both $\mathcal{P}_0$ and $\mathcal{P}_0'$ are stable under multiplication by natural numbers, the same is true of $\mathcal{P}_0 \cap \mathcal{P}_0'$.

(iii) The relative interior $\mathrm{relint}(\mathcal{C})$ of a non-empty rational cone $\mathcal{C} \subset \mathbf{R}^n$ satisfies $\mathrm{relint}(\mathcal{C}) \cap \mathbf{Q}^n \neq \varnothing$ (in fact, $\mathrm{relint}(\mathcal{C}) \cap \mathbf{Q}^n$ is dense in $\mathcal{C}$). Since $\mathrm{relint}(\mathcal{C})$ is a relatively open (hence half-open) cone, it is stable under multiplication by positive scalars. We can therefore clear denominators and conclude that $\mathrm{relint}(\mathcal{C}) \cap \mathbf{Z}^n \neq \varnothing$. Hence, if $\mathcal{P}_0 = \varnothing$, then $\mathcal{C}_0 = \varnothing$. Suppose that $\mathcal{P}_0 \neq \varnothing$. There exists $\beta \in \mathcal{P}_0 \cap \mathbf{Q}^n$ and thus $a\beta \in \mathcal{P}_0 \cap \mathbf{Z}^n = \mathcal{C}_0 \cap \mathbf{Z}^n$ for some $a \in \mathbf{N}$ whence $\mathcal{C}_0 \neq \varnothing$.

(iv) For $\omega \in \mathcal{C}_0 \cap \mathbf{Q}^n$, there exists $a \in \mathbf{N}$ with $a\omega \in \mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{P}_0 \cap \mathbf{Z}^n \subset \mathcal{P}_0$. Hence, every cone containing $\mathcal{P}_0$ also contains $\mathcal{C}_0 \cap \mathbf{Q}^n$, a dense subset of $\bar{\mathcal{C}}_0$. Similarly, $\mathcal{P}_0 \cap \mathbf{Q}^n \subset \bar{\mathcal{C}}_0$ so that $\mathcal{P}_0 \subset \bar{\mathcal{C}}_0$.

(v) By (iii), we may assume that $\mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{C}_0' \cap \mathbf{Z}^n \neq \varnothing$. Let $\mathcal{P}_0$ be a model of $\mathcal{C}_0$. Then $\mathcal{P}_0$ is also a model of both $\mathcal{C}_0'$ and $\mathcal{C}_0'' := \mathcal{C}_0 \cap \mathcal{C}_0'$. By (iv), $\mathcal{C}_0, \mathcal{C}_0', \mathcal{C}_0''$ all have the same closure, $\mathcal{C}$ say. Suppose that $\mathcal{C}_0'' \neq \mathcal{C}_0$. Then there exists a face τ of $\mathcal{C}$ such that $\mathrm{relint}(\tau) \subset \mathcal{C}_0$ but $\tau \cap \mathcal{C}_0'' = \varnothing$. Let $\omega \in \mathrm{relint}(\tau) \cap \mathbf{Z}^n$. Then $\omega \in \mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{C}_0'' \cap \mathbf{Z}^n$, a contradiction. Hence, $\mathcal{C}_0 \subset \mathcal{C}_0'$ and so $\mathcal{C}_0 = \mathcal{C}_0'$ by symmetry. $\blacklozenge$

Given a model $\mathcal{P}_0$ of $\mathcal{C}_0$, it remains to recover $\bar{\mathcal{C}}_0$ explicitly from $\mathcal{P}_0$. This can be done using the next lemma and the arguments that follow it.

Lemma 8.3. *Let $\mathcal{Q} \subset \mathbf{R}^n$ be the convex hull of the non-empty set $\{\alpha_1, \ldots, \alpha_e\} \subset \mathbf{R}^n$. Let $\mathcal{D} \subset \mathbf{R}^n$ be a cone. Write $\mathcal{C} := \mathrm{cone}(\alpha_1, \ldots, \alpha_e)$. Then $\mathcal{C} + \mathcal{D}$ is the smallest cone containing $\mathcal{Q} + \mathcal{D}$.*

Proof. Clearly, $\mathcal{C} + \mathcal{D}$ is a cone containing $\mathcal{P} := \mathcal{Q} + \mathcal{D}$. Let $\mathcal{B} \supset \mathcal{P}$ be a cone. Then $\mathcal{Q} \subset \mathcal{B}$ and hence $\mathcal{C} \subset \mathcal{B}$. Fix $\boldsymbol{x} \in \mathcal{Q}$ and let $\boldsymbol{y} \in \mathcal{D}$. Then $\boldsymbol{x} + a\boldsymbol{y} \in \mathcal{P}$ for $a > 0$ whence $a^{-1}\boldsymbol{x} + \boldsymbol{y} \in \mathcal{B}$. As $\mathcal{B}$ is closed, we conclude that $\boldsymbol{y} \in \mathcal{B}$ and thus $\mathcal{D} \subset \mathcal{B}$. ♦

Recall that (rational) polyhedra in $\mathbf{R}^n$ are exactly the sets of the form $\mathcal{Q} + \mathcal{D}$, where $\mathcal{Q} \subset \mathbf{R}^n$ is a (rational) polytope and $\mathcal{D} \subset \mathbf{R}^n$ is a (rational) cone; see e.g. [4, Thm 4.13]. Let $\mathcal{P}_0$ be a model of a non-empty rational half-open cone $\mathcal{C}_0$. Writing $\mathcal{P}_0 = \mathcal{Q} + \mathcal{D}$ for (rational) $\mathcal{Q}, \mathcal{D}$ as in Lemma 8.3, Proposition 8.2(iv) allows us to recover $\bar{\mathcal{C}}_0$.

9 Examples

We discuss examples of previously unknown topological zeta functions computed using Zeta [28]. For more examples, we refer to the database of topological subring, ideal, and submodule zeta functions included with Zeta.

9.1 Five-dimensional nilpotent Lie algebras: Fil_4

As in [36, Thm 3.6], let Fil_4 be the nilpotent Lie ring with $\mathbf{Z}$-basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_5)$ and Lie bracket $[\boldsymbol{e}_1, \boldsymbol{e}_2] = \boldsymbol{e}_3$, $[\boldsymbol{e}_1, \boldsymbol{e}_3] = \boldsymbol{e}_4$, $[\boldsymbol{e}_1, \boldsymbol{e}_4] = \boldsymbol{e}_5$, $[\boldsymbol{e}_2, \boldsymbol{e}_3] = \boldsymbol{e}_5$, and $[\boldsymbol{e}_i, \boldsymbol{e}_j] = 0$ for $i \leqslant j$ not listed above. As we explained in [27, §7.3], with the sole exception of $\mathrm{Fil}_4 \otimes_{\mathbf{Z}} \mathbf{C}$, each of the 16 isomorphism classes of non-trivial nilpotent Lie $\mathbf{C}$-algebras of dimension at most 5 admits a $\mathbf{Z}$-form whose local subring zeta functions have been computed. We can use TopologicalZetaFunction (Algorithm 4.1) and Zeta to confirm that for the 15 known types, the topological zeta function coincides with the one deduced from p-adic formulae using the informal method from the introduction. The local subring zeta functions of Fil_4 have so far resisted attempts at computing them [36, p. 57]. In [27, Eqn (7.8)], we announced that

$$
\begin{aligned}
\zeta_{\mathrm{Fil}_4,\mathrm{top}}(\boldsymbol{s}) = \big(& 392031360\boldsymbol{s}^9 - 5741480808\boldsymbol{s}^8 + 37286908278\boldsymbol{s}^7 - 140917681751\boldsymbol{s}^6 + \\
& 341501393670\boldsymbol{s}^5 - 550262853249\boldsymbol{s}^4 + 589429290044\boldsymbol{s}^3 - \\
& 404678115300\boldsymbol{s}^2 + 161557332768\boldsymbol{s} - \\
& 28569052512\big) \big/ \big(3(15\boldsymbol{s} - 26)(7\boldsymbol{s} - 12)(7\boldsymbol{s} - 13)(6\boldsymbol{s} - 11)^3 \\
& (5\boldsymbol{s} - 8)(5\boldsymbol{s} - 9)(4\boldsymbol{s} - 7)^2(3\boldsymbol{s} - 4)(2\boldsymbol{s} - 3)(\boldsymbol{s} - 1)\boldsymbol{s}\big).
\end{aligned}
\tag{9.1}
$$

For a group-theoretic interpretation, since the topological subgroup zeta function of a torsion-free, finitely generated nilpotent group G only depends on the $\mathbf{C}$-isomorphism

type of $\mathfrak{L}(G) \otimes_{\mathbf{Q}} \mathbf{C}$ (see [27, Prop. 5.19(ii)]), we thus obtain a complete classification of topological subgroup zeta functions of nilpotent groups of Hirsch length at most 5; see the database included with Zeta.

We will now provide details on the computation leading to (9.1). In doing so, we illustrate the key steps of Algorithm 4.1.

Constructing an initial toric datum. The first step is to construct Laurent polynomials as in Theorem 2.2(i) (see the end of §2) and an associated toric datum as in Remark 3.4; the input of TOPOLOGICALZETAFUNCTION (Algorithm 4.1) then consists of said toric datum, $\mathcal{T}^0$ say, and a matrix $\beta \in \mathrm{M}_{5 \times 15}(\{0, 1\})$ (which can be easily constructed as in Remark 3.4).

Using the defining basis $(e_1, \ldots, e_5)$ of Fil_4 and after performing simplification steps (see §7.2), we thus obtain the initial toric datum $\mathcal{T}^0 = (\mathcal{C}_0; f_1, f_2, f_3)$, where

$$\mathcal{C}_0 = \Big\{ (\omega_1, \ldots, \omega_{15}) \in \mathbf{R}^{15}_{\geqslant 0} : \omega_{10} \leqslant \omega_1 + \omega_6, \quad \omega_{13} \leqslant \omega_1 + \omega_{10},$$
$$\omega_{15} \leqslant \omega_6 + \omega_{10}, \quad \omega_{15} \leqslant \omega_1 + \omega_{13} \Big\}$$

and $f_1, f_2, f_3 \in \mathbf{Q}[X_1^{\pm 1}, \ldots, X_{15}^{\pm 1}]$ are given by

$$f_1 = \underline{X_2 X_{10} X_{15}^{-1} - X_1 X_{10} X_{13}^{-1} X_{14} X_{15}^{-1}} + X_1 X_{11} X_{15}^{-1}$$
$$f_2 = \underline{X_1 X_7 X_{13}^{-1} - X_1 X_6 X_{10}^{-1} X_{11} X_{13}^{-1}}$$
$$f_3 = \underline{X_1 X_7 X_{13}^{-1} X_{14} X_{15}^{-1} - X_2 X_7 X_{15}^{-1}} + X_3 X_6 X_{15}^{-1} - X_1 X_8 X_{15}^{-1} +$$
$$X_1 X_6 X_{10}^{-1} X_{12} X_{15}^{-1} - X_1 X_6 X_{10}^{-1} X_{11} X_{13}^{-1} X_{14} X_{15}^{-1};$$

we will explain the underlines below.

Balancing and regularity testing. Our next task is to decompose $\mathcal{C}_0$ using the normal cones of $\mathcal{N} := \mathrm{New}(f_1 f_2 f_3) \subset \mathbf{R}^{15}$ to obtain a system of *balanced* toric data. Using Sage, we find that $\mathcal{N}$ is a 6-dimensional polytope with 27-vertices and a total of 395 faces; note that Sage regards $\varnothing$ as a face of $\mathcal{N}$ (and hence reports a total number of 396 faces) but that we do not. For each face $\tau \subset \mathcal{N}$, we then simplify the balanced toric datum $\big(\mathcal{C}_0 \cap \mathrm{N}_\tau(\mathcal{N}); f_1, f_2, f_3 \big)$ (see §7.2) and test it for regularity (see §5.3). It turns out that all but 4 of these 395 toric data are already regular.

Singularity and reduction. We now consider one of the aforementioned four singular toric data arising from $(\mathcal{C}_0; f_1, f_2, f_3)$ in detail, namely $(\tilde{\mathcal{C}}_0; f_1, f_3)$, where

$$\tilde{\mathcal{C}}_0 = \Big\{ (\omega_1, \ldots, \omega_{15}) \in \mathbf{R}^{15}_{\geqslant 0} : \omega_1 \leqslant \omega_2 + \omega_{13}, \qquad \omega_{10} \leqslant \omega_1 + \omega_6,$$
$$\omega_{13} \leqslant \omega_1 + \omega_7, \qquad \omega_{13} \leqslant \omega_1 + \omega_{10},$$
$$\omega_{15} \leqslant \omega_1 + \omega_{13}, \qquad \omega_{15} \leqslant \omega_6 + \omega_{10},$$
$$\omega_1 + \omega_{14} = \omega_2 + \omega_{13},$$

$$\omega_2 + \omega_7 < \omega_1 + \omega_8, \qquad\qquad \omega_2 + \omega_7 < \omega_3 + \omega_6,$$
$$\omega_2 + \omega_{10} < \omega_1 + \omega_{11}, \qquad\qquad \omega_7 + \omega_{10} < \omega_6 + \omega_{11},$$
$$\left.\omega_2 + \omega_7 + \omega_{10} < \omega_1 + \omega_6 + \omega_{12}\right\},$$

which is a 14-dimensional half-open cone contained in $\mathcal{C}_0$. It is readily verified that the initial forms of f_1, f_2, and f_3 on $\tilde{\mathcal{C}}_0$ are exactly the underlined parts from above. In particular, since $\mathrm{in}_{\tilde{\mathcal{C}}_0}(f_2)$ is a term, we can see why f_2 has been discarded by the simplification step.

One checks that while $\mathrm{in}_{\tilde{\mathcal{C}}_0}(f_1) = 0$ and $\mathrm{in}_{\tilde{\mathcal{C}}_0}(f_3) = 0$ both define smooth hypersurfaces within $\mathbf{T}_{\mathbf{Q}}^{15}$, the rank condition defining regularity (see Definition 5.5) is violated precisely on the subvariety (subtorus, in fact) defined by $X_1 X_{14} = X_2 X_{13}$. Indeed, looking at the initial forms of f_1 and f_3, we see that the failure of regularity is due to these initial forms being *identical* up to multiplication by a unit in $\mathbf{Q}[X_1^{\pm 1}, \ldots, X_{15}^{\pm 1}]$; indeed, $\mathrm{in}_{\tilde{\mathcal{C}}_0}(f_1)/\mathrm{in}_{\tilde{\mathcal{C}}_0}(f_3) = -X_7^{-1} X_{10}$. The reduction procedure explained in §7.3 arose from the observation that such geometrically simple causes of singularity are remarkably common in practice.

Reduction can repair the failure of regularity of $(\tilde{\mathcal{C}}_0; f_1, f_3)$ as follows. Define terms $t_1 := X_2 X_{10} X_{15}^{-1}$ and $t_3 := -X_2 X_7 X_{15}^{-1}$ of f_1 and f_3, respectively; note that $t_1^{-1} t_3 = -X_7 X_{10}^{-1}$. Define $\tilde{\mathcal{C}}_0^{\leqslant} = \{\omega \in \tilde{\mathcal{C}}_0 : \omega_{10} \leqslant \omega_7\}$, $\tilde{\mathcal{C}}_0^{>} = \{\omega \in \tilde{\mathcal{C}}_0 : \omega_7 < \omega_{10}\}$, $g_3 = f_3 - t_1^{-1} t_3 f_1$, and $g_1 = f_1 - t_1 t_3^{-1} f_3$. Then

$$
\begin{aligned}
g_3 &= X_3 X_6 X_{15}^{-1} - X_1 X_8 X_{15}^{-1} + X_1 X_7 X_{10}^{-1} X_{11} X_{15}^{-1} + \\
&\quad X_1 X_6 X_{10}^{-1} X_{12} X_{15}^{-1} - X_1 X_6 X_{10}^{-1} X_{11} X_{13}^{-1} X_{14} X_{15}^{-1} \text{ and} \\
g_1 &= X_3 X_6 X_7^{-1} X_{10} X_{15}^{-1} - X_1 X_7^{-1} X_8 X_{10} X_{15}^{-1} + X_1 X_{11} X_{15}^{-1} + \\
&\quad X_1 X_6 X_7^{-1} X_{12} X_{15}^{-1} - X_1 X_6 X_7^{-1} X_{11} X_{13}^{-1} X_{14} X_{15}^{-1}.
\end{aligned}
$$

Setting $\mathcal{T}^{\leqslant} := (\tilde{\mathcal{C}}_0^{\leqslant}; f_1, g_3)$ and $\mathcal{T}^{>} := (\tilde{\mathcal{C}}_0^{>}; g_1, f_3)$, we obtain a partition $\{\mathcal{T}^{\leqslant}, \mathcal{T}^{>}\}$ of $(\tilde{\mathcal{C}}_0; f_1, f_3)$. As we will now explain, our particular choice of a reduction candidate (see §7.3) eliminates the source of the singularity of $(\tilde{\mathcal{C}}_0; f_1, f_3)$ that we isolated above. In general, it is possible for choices of reduction candidates to introduce new singularities. In order to verify that this is not the case here, we apply the balancing procedure from §5.2 followed by simplification to both $\mathcal{T}^{\leqslant}$ and to $\mathcal{T}^{>}$. In doing so, each of these toric data is partitioned into 31 balanced conditions, 31 being the number of faces of the Newton polytope of g_1 and of g_3. Fortunately, every single one the resulting 62 toric data is now regular, which concludes our efforts regarding $(\tilde{\mathcal{C}}_0; f_1, f_3)$.

The other three singular toric data mentioned above can be handled in a very similar way. Each of them is cut in two by reduction and each piece is then decomposed into 15 regular conditions by BALANCE. In particular, in each case, a single application of the reduction step followed by balancing and simplification immediately yields regular conditions only—this is not generally true.

After completion of the main loop in Algorithm 4.1, we have constructed a total of 543 regular toric data constituting a partition of $\mathcal{T}^0$ from above.

Final stage. It remains to apply the function EVALUATETOPOLOGICALLY to each of the 543 aforementioned toric data and to recover $\zeta_{\mathrm{Fil}_4,\mathrm{top}}(s)$ from a sum of rational functions. As it is unlikely to offer any new insights, we chose not to give details on the tedious acts of computing Euler characteristics, triangulating cones, and manipulating rational functions that constitute this step. We note that the computations documented in this section were obtained by stepping through the code of Zeta [28].

Statistics. We briefly indicate the extent to which practical applications of TOPO-LOGICALZETAFUNCTION rely on machine computations. Using Zeta on an Intel Xeon E5-2670 (8 cores) running Sage 6.7, the computation of $\zeta_{\mathrm{Fil}_4,\mathrm{top}}(s)$ sketched above took about 88 minutes in total. The main loop in TOPOLOGICALZETAFUNCTION was completed after less than 2 minutes; the vast majority of time was then spent in the final line of TOPOLOGICALZETAFUNCTION which used 16 parallel processes (see §8.3). Using polynomial interpolation, the final formula (9.1) was then recovered from a sum of 12,869,940 rational functions as explained in §8.3.

9.2 Other examples

There are various interesting examples which are similar to Fil_4 in the sense that a single application of the reduction step to singular toric data already suffices. One such example is given by the topological submodule zeta function of the full unipotent group $\mathrm{U}_5(\mathbf{Z}) \leqslant \mathrm{GL}_5(\mathbf{Z})$ acting on its natural module discussed in [27, §7.3]. For a non-nilpotent, commutative, and associative example, we find the topological subring zeta function of $\mathbf{Z}[X]/X^4$ to be

$$\zeta_{\mathbf{Z}[X]/X^4,\mathrm{top}}(s) = \frac{2021760s^5 - 8509620s^4 + 14322332s^3 - 12036071s^2 + 5044460s - 842400}{168480(6s-5)(4s-3)(s-1)^6 s}.$$

A computation of similar overall complexity which however requires multiple iterations of reduction yields the formula for $\zeta_{\mathfrak{gl}_2(\mathbf{Z}),\mathrm{top}}(s)$ announced in [27, (7.7)]. For a more complicated example, consider $\mathrm{Fil}_4 \oplus(\mathbf{Z},0)$, where $(\mathbf{Z},0)$ denotes $\mathbf{Z}$ regarded as an abelian Lie ring. After about 3.5 days (same machine as for Fil_4 above), Zeta reports that

$$\begin{aligned}
\zeta_{\mathrm{Fil}_4 \oplus(\mathbf{Z},0),\mathrm{top}}(s) = \big(&52839554826240s^{15} - 1612571385729024s^{14} + \\
&22945067840268288s^{13} - 201917310138409536s^{12} + \\
&1228942670032455984s^{11} - 5479610770178424720s^{10} + \\
&18489925054934205732s^9 - 48077179247205683304s^8 + \\
&97118269735864324559s^7 - 152405042677332499112s^6 + \\
&184268407184801648476s^5 - 168562287295854189878s^4 + \\
&112921211241642321545s^3 - 52295417007047312650s^2 + \\
&14969814525806597400s - 1996549752637440000\big)\big/
\end{aligned}$$

$$(48(15s - 31)(15s - 34)(13s - 28)(12s - 25)(9s - 20)(7s - 15)$$
$$(7s - 16)(6s - 13)^3(5s - 11)(4s - 9)^3(3s - 5)(s - 1)(s - 2)^4 s). \quad (9.2)$$

For further examples of topological zeta functions of the type considered in this article, see Zeta and the database that comes with it.

9.3 On the reliability of our computations

When it comes to trusting computer output such as the examples given above, caution is certainly warranted. Apart from possible bugs in the author's code, the sheer number of mathematical libraries and programs relied upon by Zeta is a natural source of concern.

Independent confirmation. As a simple test, we can use the many examples of local zeta functions computed by Woodward and others and compare the associated naive topological zeta functions (obtained via symbolic expansion in $p - 1$ as indicated in the introduction) with the ones obtain using Zeta, assuming our method applies. Our implementation passes this test for all examples from [15] that we considered. Conversely, our machine computations thus provide evidence for the correctness of these formulae which were often obtained using complicated, at least partially manual, and often undocumented computations.

Conjectures. For genuinely new examples such as the topological subring zeta functions of Fil_4 and $\mathrm{Fil}_4 \oplus (\mathbf{Z}, 0)$, we regard the peculiar conjectural features of topological zeta functions from [27, §8] as further evidence of the reliability of our implementation—indeed, computational errors can easily destroy these properties. The lengthy formula (9.2), for example, has all the properties predicted by the conjectures in [27, §8]. In addition, we observe that $\zeta_{\mathrm{Fil}_4,\mathrm{top}}(s)$ and $\zeta_{\mathrm{Fil}_4 \oplus (\mathbf{Z},0),\mathrm{top}}(s)$ agree at infinity in the following sense. Given any possibly non-associative ring $\mathcal{A}$ of additive rank d, let $\mathrm{m}(\mathcal{A}) := s^d \zeta_{\mathcal{A},\mathrm{top}}(s^{-1})\big|_{s=0}$. The "degree conjecture" [27, Conj. I] asserts that $0 \neq \mathrm{m}(\mathcal{A}) \neq \infty$.

Conjecture V. $\mathrm{m}(\mathcal{A}) = \mathrm{m}(\mathcal{A} \oplus (\mathbf{Z}, 0))$.

For instance, (9.1) and (9.2) show that $\mathrm{m}(\mathrm{Fil}_4) = \mathrm{m}(\mathrm{Fil}_4 \oplus (\mathbf{Z}, 0)) = 463/1350$.

The effect of the operation $\mathcal{A} \mapsto \mathcal{A} \oplus (\mathbf{Z}, 0)$ (let alone arbitrary direct sums) on subring or ideal zeta functions (local or topological) is poorly understood in general. For a few specific examples of rings $\mathcal{A}$, formulae for local zeta functions of $\mathcal{A} \oplus (\mathbf{Z}^r, 0)$ are known for all $r \geqslant 0$ (see [15]); these formulae are consistent with Conjecture V. The simple patterns exhibited by the formulae for known instances of such families seem to be exceptional, as e.g. suggested by various examples of topological zeta functions included with Zeta. The experimental evidence underpinning Conjecture V is all the more remarkable in view of the generally increased complexity of $\zeta_{\mathcal{A} \oplus (\mathbf{Z},0),\mathrm{top}}(s)$ compared with $\zeta_{\mathcal{A},\mathrm{top}}(s)$.

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