Computing topological zeta functions of groups, algebras, and modules, I

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We develop techniques for computing zeta functions associated with nilpotent groups, not necessarily associative algebras, and modules, as well as Igusa-type zeta functions. At the heart of our method lies an explicit convex-geometric formula for a class of p-adic integrals under non-degeneracy conditions with respect to associated Newton polytopes. Our techniques prove to be especially useful for the computation of topological zeta functions associated with algebras, resulting in the first systematic investigation of their properties.

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Contents

1 Introduction

Zeta functions of nilpotent groups. For a finitely generated group G, the number $a_n(G)$ of subgroups of G of index n is finite for all $n \geqslant 1$. Suppose that G is torsion-free and nilpotent. A powerful tool to analyse the arithmetic properties of the sequence

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$a_1(G), a_2(G), \ldots$ is the **subgroup zeta function** of G introduced by Grunewald, Segal, and Smith [27]. It is given by the Dirichlet series $\zeta_G(s) = \sum_{n=1}^{\infty} a_n(G)n^{-s}$ which defines an analytic function in some complex right half-plane. As a consequence of nilpotency, $\zeta_G(s)$ admits an Euler product decomposition $\zeta_G(s) = \prod_p \zeta_{G,p}(s)$ into local factors $\zeta_{G,p}(s) = \sum_{d=0}^{\infty} a_{p^d}(G) \cdot p^{-ds}$ indexed by primes p, see [27, Prop. 1.3]. A much deeper fact [27, Thm 1] is that each $\zeta_{G,p}(s)$ is a rational function in p^{-s} over $\mathbf{Q}$. By considerably advancing the techniques pioneered in [27], du Sautoy and Grunewald [20, Thm 1.1(1)] derived a fundamental result on nilpotent groups: the degree of polynomial subgroup growth of G, a priori a positive real number, is in fact rational. Their proof using zeta functions constitutes a crucial step in the development of the theory of zeta functions of groups and rings as a subject of independent interest.

Zeta functions of groups, rings, and modules. Two key insights of [27], some details of which we will recall in §2, have been central to subsequent developments in the theory of zeta functions of nilpotent groups. First, the enumeration of subgroups of such a group is essentially reduced, via linearisation, to the enumeration of subrings of an associated nilpotent Lie ring. Second, the corresponding local subring zeta functions $\zeta_{\mathcal{L},p}(s)$ of such a ring $\mathcal{L}$ admit explicit expressions in terms of p-adic integrals, thus rendering the subject amenable to a range of powerful techniques from arithmetic geometry and model theory, in particular. By invoking a result of Denef on the rationality of certain definable integrals, the first major result [27, Thm 3.5] in this direction is that each $\zeta_{\mathcal{L},p}(s)$ is rational in p^{-s} over $\mathbf{Q}$. As it turns out, many fundamental properties of $\zeta_{\mathcal{L},p}(s)$ remain intact when the requirement that $\mathcal{L}$ be a nilpotent Lie ring is relaxed whence the study of zeta functions of possibly non-associative rings evolved into an area of its own.

As we will explain in §2, the above framework also accommodates Solomon's zeta functions [47] enumerating submodules associated with integral representations; see [8] for applications of p-adic integration in this context. Finally, we note that the theory of representation growth, a presently very active branch of asymptotic group theory (see e.g. [1]), incorporates much of the machinery alluded to above.

Igusa's local zeta function. Following the introduction of techniques from p-adic integration, subsequent major advances in the theory of zeta functions of groups and rings, for example [20, 53], drew heavily upon results surrounding Igusa's local zeta function. We refer to [15, 42] for introductions to the area. Write $\mathbf{Z}_p$ for the ring of p-adic integers. In its simplest incarnation, Igusa's local zeta function $\mathsf{Z}_{f,p}(s)$ associated with a polynomial $f \in \mathbf{Z}_p[X_1, \ldots, X_n]$ is given by $\mathsf{Z}_{f,p}(s) = \int_{\mathbf{Z}_p^n} |f(\boldsymbol{x})|^s \,\mathrm{d}\mu(\boldsymbol{x})$, where $|\cdot|$ denotes the usual p-adic absolute value, μ is the additive Haar measure on $\mathbf{Z}_p^n$ with $\mu(\mathbf{Z}_p^n) = 1$, and $s \in \mathbf{C}$ satisfies $\mathrm{Re}(s) > 0$. Igusa'a local zeta function $\mathsf{Z}_{f,p}(s)$ enumerates solutions of congruences $f(\boldsymbol{x}) \equiv 0 \bmod p^d$ via

$$\frac{1 - p^{-s}\mathsf{Z}_{f,p}(s)}{1 - p^{-s}} = \sum_{d=0}^{\infty} \#\{\bar{\boldsymbol{x}} \in (\mathbf{Z}/p^d)^n : f(\bar{\boldsymbol{x}}) \equiv 0 \bmod p^d\} \cdot p^{-(s+n)d}.$$

Igusa proved (see [29, Thm 8.2.1]) that $\mathsf{Z}_{f,p}(s)$ is always rational in p^{-s} over $\mathbf{Q}$. For a polynomial $f \in \mathbf{Z}[X_1,\dots,X_n]$, we may consider $\mathsf{Z}_{f,p}(s)$ for all primes p. Denef [13] gave a formula for $\mathsf{Z}_{f,p}(s)$ that, in particular, explains the dependence on the prime p. A fundamental insight of his was that $\mathsf{Z}_{f,p}(s)$ can, for almost all p, be expressed explicitly in terms of an embedded resolution of singularities of $\mathrm{Spec}(\mathbf{Q}[X_1,\dots,X_n]/f) \subset \mathbf{A}^n_{\mathbf{Q}}$.

It is evident from Igusa's rationality result what it means to compute $\mathsf{Z}_{f,p}(s)$. The existence of algorithms for resolution of singularities in characteristic zero [5,52] implies that Denef's approach can, at least in principle, be used to carry out such computations. However, the rather unfavourable complexity estimates of known algorithms for resolving singularities (cf. [4]) as well as practical experience show that this approach is only useful in very small dimensions—in fact, often only for $n \leqslant 3$.

Among the more practically-minded methods for computing $\mathsf{Z}_{f,p}(s)$, a central place is taken by "toroidal" ones that, under suitable non-degeneracy assumptions, yield explicit formulae for $\mathsf{Z}_{f,p}(s)$ in terms of the Newton polytope (or Newton polyhedron) of f; see [16,51] and the references in §4.4. Such methods go back to work of Khovanskii [33] and others [37,49] on "toroidal compactifications" of complex varieties. These results have been a major source of inspiration for the work described in the present article.

Results I: non-degenerate p-adic integrals. In Sections 3–4, we study multivariate local zeta functions defined by p-adic integrals attached to globally defined collections of polynomials. The rather general class of integrals that we consider specialises to local subring, subgroup, and submodule zeta functions as well as to various Igusa-type zeta functions. Our main result (Theorem 4.10) is an explicit convex-geometric formula for the aforementioned integrals under non-degeneracy assumptions of the defining polynomials with respect to their Newton polytopes. When applied in the context of Igusa-type zeta functions, our formula generalises a result of Denef and Hoornaert [16] and provides an alternative to work of Veys and Zúñiga-Galindo [51]. To the author's knowledge, our formula is the first of its kind that applies to subring and submodule zeta functions.

Topological zeta functions. The **topological zeta function** $\mathsf{Z}_{f,\mathrm{top}}(s) \in \mathbf{Q}(s)$ associated with a polynomial $f \in \mathbf{Z}[X_1,\dots,X_n]$ was introduced by Denef and Loeser [17]. It can be thought of as a limit "$p \to 1$" of the local zeta functions $\mathsf{Z}_{f,p}(s)$ from above. For example, it is easy to see that $\mathsf{Z}_{X,p}(s) = \int_{\mathbf{Z}_p} \lvert x\rvert^s \,\mathrm{d}\mu(x) = \frac{1-p^{-1}}{1-p^{-s-1}}$. Informally, we obtain $\mathsf{Z}_{X,\mathrm{top}}(s)$ as the constant term of $\mathsf{Z}_{X,p}(s)$, expanded as a series in $p-1$. In our example, disregarding questions of convergence, using the binomial series, we find that $\mathsf{Z}_{X,p}(s) = 1/(s+1) + \mathcal{O}(p-1)$ whence $\mathsf{Z}_{X,\mathrm{top}}(s) = 1/(s+1)$. While our informal approach fails to be rigorous in a number of ways, the results of [17] serve to fill these gaps. Using a different approach, Denef and Loeser [18] later obtained $\mathsf{Z}_{f,\mathrm{top}}(s)$ as a specialisation of the motivic zeta function associated with f, another invention of theirs. As already observed in [17], topological zeta functions associated with polynomials retain interesting properties of their more complicated p-adic ancestors. At the same time, they tend to be more amenable to computations. Important contributions to their study have, in particular, been made by Veys and his collaborators, see e.g. [38,39,50].

In [21], du Sautoy and Loeser introduced motivic subalgebra zeta functions. As a by-product, they defined associated topological zeta functions and gave a number of examples in cases where the motivic zeta function has been computed. Informally, if $\mathcal{L}$ is a possibly non-associative ring of additive rank d, then its topological subring zeta function $\zeta_{\mathcal{L},\mathrm{top}}(s)$ is the constant term of $(1-p^{-1})^d\zeta_{\mathcal{L},p}(s)$ as a series in $p-1$. For instance, if $(\mathbf{Z}^d, 0)$ denotes $\mathbf{Z}^d$ endowed with the zero multiplication, then, as is well-known, $\zeta_{(\mathbf{Z}^d,0),p}(s) = \zeta_p(s)\zeta_p(s-1)\cdots\zeta_p(s-(d-1))$, where $\zeta_p(s) = 1/(1-p^{-s})$ is the p-local factor of the Riemann zeta function. We thus expect $\zeta_{(\mathbf{Z}^d,0),\mathrm{top}}(s) = \frac{1}{s(s-1)\cdots(s-(d-1))}$ which is indeed the case using the definition of $\zeta_{\mathcal{L},\mathrm{top}}(s)$ given in the present article (but see Remark 5.18). For a rigorous treatment, in §5, we give a self-contained introduction to topological zeta functions, including those arising from the enumeration of subrings, based on the original approach from [17].

Results II: computing topological zeta functions. In §6, we continue our investigation of the p-adic integrals in §4 from the point of view of topological zeta functions. As our main result (Theorem 6.7), we give explicit and purely combinatorial formulae for the topological zeta functions associated with our integrals under non-degeneracy conditions.

In §7, we illustrate our formulae by explicitly computing examples of zeta functions of groups, rings, and modules, old and new. While such computations can be carried out by hand in small cases, the true strength of our approach (the topological part, in particular) lies in its machine-friendly form. Indeed, based on Theorem 6.7, the author has developed a practical method for computing topological zeta functions associated with nilpotent groups, not necessarily associative rings, modules, as well as various Igusa-type zeta functions under non-degeneracy assumptions. A detailed account of the computational techniques and extensions needed to transform the present theoretical work into such a practical method is given in [44]. As an illustration, our method allows us to compute the previously unknown topological subring zeta function of the nilpotent Lie ring Fil_4, see (7.8). In contrast, according to [24, §2.13], the p-adic subring zeta functions of Fil_4 have so far resisted "repeated efforts" to compute them. As explained in §7.3, this particular example concludes the determination of the topological subgroup zeta functions of all nilpotent groups of Hirsch length at most 5.

Based on considerable experimental evidence, in §8, we state a number of intriguing conjectures. It would seem that some of these conjectures are in fact topological shadows of conjectural properties of p-adic zeta functions that went previously unnoticed. For example, if $\mathcal{L}$ is a nilpotent Lie ring of additive rank d, then Conjecture IV predicts that the meromorphic continuations of $\zeta_{\mathcal{L},p}(s)$ and $\zeta_{(\mathbf{Z}^d,0),p}(s)$ (where $(\mathbf{Z}^d, 0)$ denotes $\mathbf{Z}^d$ endowed with the zero multiplication) agree at $s = 0$ in the sense that $\left.\frac{\zeta_{\mathcal{L},p}(s)}{\zeta_{(\mathbf{Z}^d,0),p}(s)}\right|_{s=0} = 1$.

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topological zeta functions.

Notation

We let "$\subset$" signify not necessarily proper inclusion. The cardinality of a set A is denoted by $\#A$ or $|A|$. We write $\mathbf{N} = \{1, 2, \dots\}$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The ideal generated by a set or family S within a given ambient ring is denoted by $\langle S \rangle$. For a family $\boldsymbol{f} = (f_i)_{i \in I}$ of polynomials, we write $\boldsymbol{f}(\boldsymbol{x}) = \big(f_i(\boldsymbol{x})\big)_{i \in I}$ and similarly for sets of polynomials. We often write $\mathbf{1} = (1, \dots, 1)$. Base change is usually denoted using subscripts. We let $\mathbf{T}^n = \operatorname{Spec}\big(\mathbf{Z}[\lambda_1^{\pm 1}, \dots, \lambda_n^{\pm 1}]\big)$ and we identify $\mathbf{T}^n(R) = (R^\times)^n$. Given a valuation ν on a field K, we write $\nu(\boldsymbol{x}) = (\nu(x_1), \dots, \nu(x_n))$ for $\boldsymbol{x} \in K^n$.

2 Global and local zeta functions of algebras and modules

In this section, we elaborate on the relationship between local zeta functions of groups and rings and p-adic integration.

2.1 Subalgebra and submodule zeta functions

Let R be the ring of integers in a global or a non-Archimedean local field of characteristic zero. We consider slight generalisations of subring [27] and submodule [47] zeta functions.

Definition 2.1.

(i) Let $\mathcal{A}$ be a not necessarily associative R-algebra. Suppose that the underlying R-module of $\mathcal{A}$ is free of finite rank. The **subalgebra zeta function** of $\mathcal{A}$ is

$$\zeta_{\mathcal{A}}(s) = \sum_{n=1}^{\infty} \#\{\mathcal{U} : \mathcal{U} \text{ is an } R\text{-subalgebra of } \mathcal{A} \text{ with } |\mathcal{A} : \mathcal{U}| = n\} \cdot n^{-s},$$

where $|\mathcal{A} : \mathcal{U}|$ denotes the cardinality of the R-module $\mathcal{A}/\mathcal{U}$.

(ii) Let M be a free R-module of finite rank and let $\mathcal{E}$ be a subalgebra of the associative R-algebra $\operatorname{End}_R(M)$. The **submodule zeta function** of $\mathcal{E}$ acting on M is

$$\zeta_{\mathcal{E} \curvearrowright M}(s) = \sum_{n=1}^{\infty} \#\{U : U \text{ is an } (\mathcal{E} + R1_M)\text{-submodule of } M \text{ with } |M : U| = n\} \cdot n^{-s}.$$

Remark 2.2.

(i) The subalgebra zeta function specialises to the global ($R = \mathbf{Z}$) and local ($R = \mathbf{Z}_p$) subring zeta functions from [27, §3]. If $R = \mathbf{Z}$ and $\mathcal{E} \otimes \mathbf{Q}$ is semisimple over $\mathbf{Q}$, then $\zeta_{\mathcal{E} \curvearrowright M}(s)$ is the zeta function introduced by Solomon in [47]; it admits an Euler factorisation and $\mathcal{E} \otimes \mathbf{Z}_p$ acting on $M \otimes \mathbf{Z}_p$ gives rise to its local factors.

(ii) The **ideal zeta function**

$$\zeta^{\triangleleft}_{\mathcal{A}}(s) := \sum_{n=1}^{\infty} \#\{\mathcal{U} \triangleleft \mathcal{A} : |\mathcal{A} : \mathcal{U}| = n\} \cdot n^{-s}$$

of $\mathcal{A}$ defined in [27, §3] (for $R = \mathbf{Z}$), itself a generalisation of the Dedekind zeta function of a number field, is an instance of a submodule zeta function—indeed, $\zeta^{\triangleleft}_{\mathcal{A}}(s) = \zeta_{\mathcal{E} \curvearrowright \mathcal{A}}(s)$ where $\mathcal{E}$ is the algebra generated by all R-module endomorphisms $x \mapsto ax$ and $x \mapsto xa$ of $\mathcal{A}$.

(iii) Since the number of all R-submodules of R^d of index n is bounded polynomially as a function of n, each of the formal Dirichlet series above defines an analytic function in some complex right half-plane, a property that is in fact enjoyed by a much larger class of rings R, see [45].

By a **prime** of a number field, we always mean a non-zero prime ideal of its ring of integers. Using primary decomposition over Dedekind domains, the Euler product factorisations in [27, 47] take the following form in the present setting.

Lemma 2.3. *Let $\mathfrak{o}$ be the ring of integers of a number field k. Let $\mathcal{A}$, M, and $\mathcal{E}$ be as in Definition 2.1 with $R = \mathfrak{o}$. Then $\zeta_{\mathcal{A}}(s) = \prod_{\mathfrak{p}} \zeta_{\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}}(s)$ and $\zeta_{\mathcal{E} \curvearrowright M}(s) = \prod_{\mathfrak{p}} \zeta_{\mathcal{E} \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}} \curvearrowright M \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}}(s)$, the products being taken over the primes of k.* ♦

Here, $\mathfrak{o}_{\mathfrak{p}}$ denotes the completion of the localisation of $\mathfrak{o}$ at $\mathfrak{p}$ or, equivalently, the valuation ring of the $\mathfrak{p}$-adic completion $k_{\mathfrak{p}}$ of k, and we regarded $\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$ and $\mathcal{E} \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$ as $\mathfrak{o}_{\mathfrak{p}}$-algebras and $M \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$ as an $\mathfrak{o}_{\mathfrak{p}}$-module.

2.2 Zeta functions of nilpotent groups: linearisation

The **normal subgroup zeta function** $\zeta^{\triangleleft}_G(s)$ of a finitely generated torsion-free nilpotent group G is defined similar to $\zeta_G(s)$ from the introduction by enumerating normal subgroups of finite index of G. The function $\zeta^{\triangleleft}_G(s)$ shares many properties with $\zeta_G(s)$, in particular the existence of canonical Euler products and rationality of local factors.

Theorem 2.4 ([27, §4])**.** *Let G be a finitely generated torsion-free nilpotent group and let $\mathfrak{L}(G)$ be the nilpotent $\mathbf{Q}$-Lie algebra attached to G via the Mal'cev correspondence. Let $\mathcal{L} \subset \mathfrak{L}(G)$ be an arbitrary Lie subalgebra over $\mathbf{Z}$ which is finitely generated as a $\mathbf{Z}$-module and which spans $\mathfrak{L}(G)$ over $\mathbf{Q}$. Then $\zeta_{G,p}(s) = \zeta_{\mathcal{L},p}(s)$ and $\zeta^{\triangleleft}_{G,p}(s) = \zeta^{\triangleleft}_{\mathcal{L},p}(s)$ for almost all primes p.*

Throughout this article, we will ignore finite sets of exceptional primes. From our point of view, the study of local zeta functions of nilpotent groups is thus subsumed by the cases of algebras and modules. Passing from nilpotent groups to $\mathbf{Z}$-algebras allows us to perform base changes to finite extensions of $\mathbf{Z}$ and $\mathbf{Z}_p$. This entirely natural operation on the level of algebras and modules will play a central role throughout this article.

2.3 Local zeta functions and $\mathfrak{P}$-adic integration

The following notational conventions will be used at various points in this article.

Notation 2.5. For a non-Archimedean local field K, let $\mathfrak{O}$ be the valuation ring and q be the residue field size of K. Let $\mathfrak{P}$ be the maximal ideal of $\mathfrak{O}$. Choose a uniformiser π and let ν be the valuation on K with $\nu(\pi) = 1$. Let $|\cdot|$ be the absolute value on K with $|x| = q^{-\nu(x)}$. For a non-empty set or family $M \subset K$, we write $\|M\| = \sup(|x| : x \in M)$. Let μ be the Haar measure on K normalised such that $\mu(\mathfrak{O}) = 1$; we use the same symbol to denote the product measure on K^n.

We often use subscripts to denote base change. Let $\mathrm{Tr}_d(R)$ be the ring of upper triangular $d \times d$-matrices over a ring R.

Theorem 2.6 ([20, §5])**.** *Let k be a number field with ring of integers $\mathfrak{o}$.*

(i) Let $\mathcal{A}$ be a not necessarily associative $\mathfrak{o}$-algebra which is free of rank d as an $\mathfrak{o}$-module. Then there exists a finite family $\boldsymbol{f} = (f_i)_{i\in I}$ of non-zero Laurent polynomials $f_i \in \mathfrak{o}[X_{ij}^{\pm 1} : 1 \leqslant i \leqslant j \leqslant d]$ with the following property: if $K \supset k$ is a p-adic field, then

$$\zeta_{\mathcal{A}_{\mathfrak{O}}}(s) = (1-q^{-1})^{-d} \int\limits_{\{\boldsymbol{x}\in \mathrm{Tr}_d(\mathfrak{O}) : \prod\limits_{1\leqslant i\leqslant j\leqslant d} x_{ij}\neq 0,\ \|\boldsymbol{f}(\boldsymbol{x})\|\leqslant 1\}} |x_{11}|^{s-1} \cdots |x_{dd}|^{s-d} \,\mathrm{d}\mu(\boldsymbol{x}), \tag{2.1}$$

where we identified $\mathrm{Tr}_d(K) \approx K^{\binom{d+1}{2}}$ and followed the conventions in Notation 2.5.

(ii) Let $\mathcal{E}$ be an associative $\mathfrak{o}$-subalgebra of $\mathrm{End}_{\mathfrak{o}}(M)$, where M is a free $\mathfrak{o}$-module of rank d. Then there are Laurent polynomials as in (i) (but usually different from those) such that the conclusion of (i) holds for $\zeta_{\mathcal{E}_{\mathfrak{O}} \curvearrowright M_{\mathfrak{O}}}(s)$ in place of $\zeta_{\mathcal{A}_{\mathfrak{O}}}(s)$.

Remark 2.7.

(i) Strictly speaking, the proof given in [20] (which relies on [27, Prop. 3.1]) only covers the case that $\mathcal{A}$ is a Lie ring over $\mathbf{Z}$ and $K = \mathbf{Q}_p$; however, the same arguments carry over, essentially verbatim, to the present setting.

(ii) For the benefit of the reader who wishes to work through the explicit examples in §7, we recall how to find a suitable $\boldsymbol{f}$ in (i), part (ii) being similar. Thus, by choosing a basis, we may identify $\mathcal{A}$ and $\mathfrak{o}^d$ as $\mathfrak{o}$-modules. Let $R := \mathfrak{o}[X_{ij} : 1 \leqslant i \leqslant j \leqslant d]$ and let $C := [X_{ij}]_{i\leqslant j} \in \mathrm{Tr}_d(R)$ with rows $C_1, \ldots, C_d$. Now let $\boldsymbol{f}$ consist of all non-zero components of all d-tuples $\det(C)^{-1}(C_m C_n)\operatorname{adj}(C)$ for $1 \leqslant m, n \leqslant d$, where the products $C_m C_n$ are taken in $\mathcal{A}_R$ and $\operatorname{adj}(C)$ denotes the adjugate matrix of C. We note that $\boldsymbol{f}$ in general very much depends on our choice of an $\mathfrak{o}$-basis of $\mathcal{A}$. Moreover, the description of $\boldsymbol{f}$ given here is often highly redundant. For actual computations such as those in §7, we tacitly apply some elementary simplifications.

Keeping the notation of Theorem 2.6, by adapting and extending a result due to Denef [13, Thm 3.1], du Sautoy and Grunewald [20, §§2–3] (see also §5.4) derived an explicit formula for $\zeta_{\mathcal{A}_{\mathfrak{O}}}(s)$ or $\zeta_{\mathcal{E}_{\mathfrak{O}} \curvearrowright M_{\mathfrak{O}}}(s)$, respectively, in terms of numerical data extracted from an embedded resolution of singularities. As in the case of Igusa's local zeta function, this approach is primarily of theoretical interest due to the infeasibility of constructing a resolution of singularities in practice. Our first major goal, to be accomplished in Theorem 4.10, is to find more practical means of computing integrals such as those in Theorem 2.6 under suitable non-degeneracy assumptions.

3 Zeta functions associated with cones and polytopes

Let K be a p-adic field with associated objects as in Notation 2.5. For a polytope $\mathcal{P} \subset \mathbf{R}^n$ and $\boldsymbol{x} \in (K^\times)^n$, we write $\mathcal{P}(\boldsymbol{x}) = \{\boldsymbol{x}^\alpha : \alpha \in \mathcal{P} \cap \mathbf{Z}^n\}$, where $\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. For $\boldsymbol{x} \in K^n$, write $\nu(\boldsymbol{x}) := (\nu(x_1), \dots, \nu(x_n))$. Note that $\nu(\boldsymbol{x}^\omega) = \langle \nu(\boldsymbol{x}), \omega \rangle$ for $\omega \in \mathbf{Z}^n$ and $\boldsymbol{x} \in (K^\times)^n$, where $\langle \cdot\,, \cdot \rangle$ denotes the standard inner product. Let $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$ be a half-open rational cone and let $\mathcal{P}_1, \dots, \mathcal{P}_m \subset \mathbf{R}^n_{\geqslant 0}$ be lattice polytopes. In this section, we give an explicit convex-geometric formula (see Proposition 3.9) for the "zeta function"

$$\int\limits_{\{\boldsymbol{x} \in K^n : \nu(\boldsymbol{x}) \in \mathcal{C}_0\}} \|\mathcal{P}_1(\boldsymbol{x})\|^{s_1} \cdots \|\mathcal{P}_m(\boldsymbol{x})\|^{s_m} \, \mathrm{d}\mu(\boldsymbol{x}), \tag{3.1}$$

where $s_1, \dots, s_m \in \mathbf{C}$ with $\operatorname{Re}(s_j) \geqslant 0$. Zeta functions of this shape constitute the building blocks of our explicit formulae for more complicated $\mathfrak{P}$-adic integrals in §4. They also generalise Igusa-type zeta functions associated with monomial ideals previously considered in the literature. Indeed, if $\mathcal{C}_0 = \mathbf{R}^n_{\geqslant 0}$, $m = 1$, and $\mathcal{P}_1 = \operatorname{conv}(\alpha_1, \dots, \alpha_r)$ for $\alpha_1, \dots, \alpha_r \in \mathbf{N}_0^n$, then (3.1) coincides with the zeta function associated with the ideal, I say, generated by $\boldsymbol{X}^{\alpha_1}, \dots, \boldsymbol{X}^{\alpha_r}$ over $\mathfrak{o}$ in the sense of [28]. In this special case, our Proposition 3.9 is analogous to [28, Prop. 2.1], the main difference being that in [28], the polytope $\mathcal{P}_1$ is replaced by the polyhedron $\operatorname{conv}\bigl(\alpha \in \mathbf{N}_0^n : \boldsymbol{X}^\alpha \in I\bigr) \subset \mathbf{R}^n_{\geqslant 0}$; cf. §4.4.

3.1 Background on cones and their generating functions

We summarise some standard material; for details see e.g. [2, 3, 11, 56].

Cones. By a **cone** in $\mathbf{R}^n$, we mean a polyhedral cone, i.e. a finite intersection of closed linear half-spaces in $\mathbf{R}^n$. If such a collection of half-spaces can be chosen to be defined over $\mathbf{Q}$, then the cone is **rational**. Equivalently, cones in $\mathbf{R}^n$ are precisely the sets of the form $\operatorname{cone}(P) := \{\sum_{\varrho \in P} \lambda(\varrho)\varrho : \lambda(\varrho) \in \mathbf{R}_{\geqslant 0}\}$ for finite $P \subset \mathbf{R}^n$. A cone

$\mathcal{C} \subset \mathbf{R}^n$ is rational if and only if $\mathcal{C} = \operatorname{cone}(P)$ for some finite $P \subset \mathbf{Z}^n$. The **dual cone** $\mathcal{P}^* := \{\omega \in \mathbf{R}^n : \langle \alpha, \omega \rangle \geqslant 0 \text{ for all } \alpha \in \mathcal{P}\}$ of a polytope $\mathcal{P} \subset \mathbf{R}^n$ is a cone in our sense. A cone $\mathcal{C} \subset \mathbf{R}^n$ is **pointed** if it does not contain any non-trivial linear subspace of $\mathbf{R}^n$. The following notion appears less frequently in the literature. Thus, a (relatively) **half-open cone** in $\mathbf{R}^n$ is a set of the form $\mathcal{C} \setminus (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r)$, where $\mathcal{C} \subset \mathbf{R}^n$ is a cone and $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are faces of $\mathcal{C}$. Equivalently, half-open cones are finite intersections of closed linear half-spaces and open ones. We say that a half-open cone is rational if it is either empty or if its closure is a rational cone. A **relatively open cone** is a non-empty half-open cone which coincides with the relative interior of its closure within the ambient Euclidean space.

Polytopes and fans. For a non-empty polytope $\mathcal{P} \subset \mathbf{R}^n$ and $\omega \in \mathbf{R}^n$, let $\operatorname{face}_\omega(\mathcal{P})$ denote the face of $\mathcal{P}$ where the function $\mathcal{P} \to \mathbf{R}, \alpha \mapsto \langle \alpha, \omega \rangle$ attains its minimum; we do not regard $\varnothing$ as a face of $\mathcal{P}$. If $\mathcal{Q} \subset \mathbf{R}^n$ is another non-empty polytope, then $\operatorname{face}_\omega(\mathcal{P} + \mathcal{Q}) = \operatorname{face}_\omega(\mathcal{P}) + \operatorname{face}_\omega(\mathcal{Q})$. The (inner, relatively open) **normal cone** of a face $\tau \subseteq \mathcal{P}$ is $\mathrm{N}_\tau(\mathcal{P}) = \{\omega \in \mathbf{R}^n : \operatorname{face}_\omega(\mathcal{P}) = \tau\}$. We have $n = \dim(\tau) + \dim(\mathrm{N}_\tau(\mathcal{P}))$. It is well-known that $\mathbf{R}^n = \bigcup_\tau \mathrm{N}_\tau(\mathcal{P})$ is a partition of $\mathbf{R}^n$ into relatively open cones. The set $\{\overline{\mathrm{N}_\tau(\mathcal{P})} : \tau \text{ is a face of } \mathcal{P}\}$ constitutes a fan, called the (inner) **normal fan** of $\mathcal{P}$.

Visibility. The following terminology is non-standard.

Definition 3.1. Let $\mathcal{C}_0 \subset \mathbf{R}^n$ be a half-open cone and let $\mathcal{P} \subset \mathbf{R}^n$ be a non-empty polytope. We say that a face $\tau \subset \mathcal{P}$ is **$\mathcal{C}_0$-visible** if $\mathrm{N}_\tau(\mathcal{P}) \cap \mathcal{C}_0 \neq \varnothing$.

Lemma 3.2. *Let $\mathcal{C} \subset \mathbf{R}^n$ be a full-dimensional cone and let $\mathcal{P} \subset \mathbf{R}^n$ be a non-empty polytope. Write $\operatorname{int}(\mathcal{C})$ for the interior of $\mathcal{C}$ and $\mathcal{C}^*$ for the dual cone of $\mathcal{C}$. Then the $\operatorname{int}(\mathcal{C})$-visible faces of $\mathcal{P}$ are precisely the non-empty compact faces of $\mathcal{P} + \mathcal{C}^*$.*

Proof. For $\omega \in \mathbf{R}^n$, we have $\operatorname{face}_\omega(\mathcal{P} + \mathcal{C}^*) = \operatorname{face}_\omega(\mathcal{P}) + \operatorname{face}_\omega(\mathcal{C}^*)$ so $\operatorname{face}_\omega(\mathcal{P} + \mathcal{C}^*)$ is non-empty and compact if and only if $\operatorname{face}_\omega(\mathcal{C}^*) = \{0\}$. By [11, Ex. 1.2.2(a)], the latter condition is equivalent to $\omega \in \operatorname{int}(\mathcal{C})$. ♦

Generating functions. Let $\lambda_1, \ldots, \lambda_n$ be algebraically independent over $\mathbf{Q}$ and write $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let $\mathbf{T}^n = \operatorname{Spec}(\mathbf{Z}[\lambda_1^{\pm 1}, \ldots, \lambda_n^{\pm 1}])$. Unless otherwise mentioned, by a commutative ring or algebra, we always mean an associative, commutative, and unital ring or algebra, respectively. Given a commutative ring R, we identify $\mathbf{T}^n(R) = (R^\times)^n$.

Definition 3.3. For a rational cone $\mathcal{C} \subset \mathbf{R}^n$, let

$$\mathrm{U}(\mathcal{C}) := \{\boldsymbol{x} \in \mathbf{T}^n(\mathbf{C}) : |\boldsymbol{x}^\omega| < 1 \text{ for } 0 \neq \omega \in \mathcal{C} \cap \mathbf{Z}^n\}.$$

Note that if $\mathcal{C} = \operatorname{cone}(\varrho_1, \ldots, \varrho_r)$ for $0 \neq \varrho_i \in \mathbf{Z}^n$, then each $\omega \in \mathcal{C} \cap \mathbf{Z}^n$ is a $\mathbf{Q}_{\geqslant 0}$-linear combination of the ϱ_i whence $\mathrm{U}(\mathcal{C}) = \{\boldsymbol{x} \in \mathbf{T}^n(\mathbf{C}) : |\boldsymbol{x}^{\varrho_i}| < 1 \text{ for } i = 1, \ldots, r\}$ follows.

The next result is well-known.

Theorem 3.4 (Cf. [3, Ch. 13]). *Let $\mathcal{C} \subset \mathbf{R}^n$ be a pointed rational cone and let $\mathcal{C}_0 \subset \mathcal{C}$ be a half-open cone with $\overline{\mathcal{C}_0} = \mathcal{C}$.*

(i) $\mathrm{U}(\mathcal{C})$ *is a non-empty open subset of* $\mathbf{T}^n(\mathbf{C})$.

(ii) $\sum_{\omega \in \mathcal{C}_0 \cap \mathbf{Z}^n} \boldsymbol{x}^\omega$ *converges absolutely and compactly on* $\mathrm{U}(\mathcal{C})$.

(iii) There exists a unique $|\mathcal{C}_0| \in \mathbf{Q}(\boldsymbol{\lambda})$ *with* $|\mathcal{C}_0|(\boldsymbol{x}) = \sum_{\omega \in \mathcal{C}_0 \cap \mathbf{Z}^n} \boldsymbol{x}^\omega$ *for* $\boldsymbol{x} \in \mathrm{U}(\mathcal{C})$.

Using the inclusion-exclusion principle, Theorem 3.4 reduces to the case $\mathcal{C}_0 = \mathcal{C}$, the situation usually considered in the literature. Let $\mathbf{Q}[\mathcal{C} \cap \mathbf{Z}^n]$ denote the $\mathbf{Q}$-subalgebra of $\mathbf{Q}(\boldsymbol{\lambda})$ spanned (over $\mathbf{Q}$) by the Laurent monomials $\boldsymbol{\lambda}^\omega$ for $\omega \in \mathcal{C} \cap \mathbf{Z}^n$. Then, as is well-known, $|\mathcal{C}|(\boldsymbol{\lambda})$ can be written as a finite sum of rational functions of the form $f(\boldsymbol{\lambda})/\prod_{j=1}^d (1 - \boldsymbol{\lambda}^{\alpha_j})$, where $f(\boldsymbol{\lambda}) \in \mathbf{Q}[\mathcal{C} \cap \mathbf{Z}^n]$, $d \leqslant \dim(\mathcal{C})$, and $0 \neq \alpha_j \in \mathcal{C} \cap \mathbf{Z}^n$. For the sake of completeness, we set $|\varnothing|(\boldsymbol{\lambda}) = 0$.

3.2 Monomial substitutions

Let $A \in \mathrm{M}_{n \times m}(\mathbf{Z})$. We also let A denote the induced linear map $\mathbf{R}^n \to \mathbf{R}^m$ acting by right-multiplication on row vectors. We further obtain an induced ring homomorphism $(-)^A \colon \mathbf{Z}[\lambda_1^{\pm 1}, \dots, \lambda_n^{\pm 1}] \to \mathbf{Z}[\xi_1^{\pm 1}, \dots, \xi_m^{\pm 1}]$ given by $(\boldsymbol{\lambda}^\alpha)^A = \boldsymbol{\xi}^{\alpha A}$ for $\alpha \in \mathbf{Z}^n$ and an induced morphism $A(-) \colon \mathbf{T}^m \to \mathbf{T}^n$ characterised by $f(A\boldsymbol{y}) = f^A(\boldsymbol{y})$ for $f \in \mathbf{Z}[\boldsymbol{\lambda}^{\pm 1}]$ and $\boldsymbol{y} \in \mathbf{T}^m(R)$, where R is a commutative ring. Thus, if $A_1, \dots, A_n$ denote the rows of A, then $A\boldsymbol{y} = (\boldsymbol{y}^{A_1}, \dots, \boldsymbol{y}^{A_n})$ for $\boldsymbol{y} \in \mathbf{T}^m(R)$.

Let $\mathcal{C} \subset \mathbf{R}^n$ be a pointed rational cone and suppose that $\mathcal{C} \cap \mathrm{Ker}(A) = \{0\}$. The image $\mathcal{C}A \subset \mathbf{R}^m$ of $\mathcal{C}$ under A is then again a pointed rational cone. The map $(-)^A \colon \mathbf{Q}[\boldsymbol{\lambda}^{\pm 1}] \to \mathbf{Q}[\boldsymbol{\xi}^{\pm 1}]$ extends to a $\mathbf{Q}$-algebra homomorphism $(-)^A \colon \mathcal{B} \to \mathcal{B}'$, where $\mathcal{B}$ is the $\mathbf{Q}$-algebra generated by $\mathbf{Q}[\boldsymbol{\lambda}^{\pm 1}]$ and all $(1 - \boldsymbol{\lambda}^\omega)^{-1}$ with $0 \neq \omega \in \mathcal{C} \cap \mathbf{Z}^n$ and $\mathcal{B}'$ is generated by $\mathbf{Q}[\boldsymbol{\xi}^{\pm 1}]$ and all $(1 - \boldsymbol{\xi}^{\omega A})^{-1}$, again for $0 \neq \omega \in \mathcal{C} \cap \mathbf{Z}^n$. In particular, $|\mathcal{C}|^A = |\mathcal{C}|(\boldsymbol{\xi}^{A_1}, \dots, \boldsymbol{\xi}^{A_n})$ is a well-defined element of $\mathcal{B}' \subset \mathbf{Q}(\boldsymbol{\xi})$.

Lemma 3.5. *Let $\mathcal{C} \subset \mathbf{R}^n$ be a pointed rational cone and let $\mathcal{C}_0 \subset \mathcal{C}$ be a half-open cone with $\overline{\mathcal{C}_0} = \mathcal{C}$. Let $A \in \mathrm{M}_{n \times m}(\mathbf{Z})$ with $\mathcal{C} \cap \mathrm{Ker}(A) = \{0\}$. Then $\sum_{\omega \in \mathcal{C}_0 \cap \mathbf{Z}^n} \boldsymbol{y}^{\omega A}$ converges absolutely and compactly on $\mathrm{U}(\mathcal{C}A)$ (see Definition 3.3). The resulting function $\mathrm{U}(\mathcal{C}A) \to \mathbf{C}$ is given by $\boldsymbol{y} \mapsto |\mathcal{C}_0|^A(\boldsymbol{y})$.*

Proof. If $\mathcal{C} = \mathrm{cone}(\varrho_1, \dots, \varrho_r)$ for $0 \neq \varrho_i \in \mathbf{Z}^n$, then $\mathcal{C}A = \mathrm{cone}(\varrho_1 A, \dots, \varrho_r A)$ and the $\varrho_i A$ are non-zero too. As $\boldsymbol{y}^{\omega A} = (A\boldsymbol{y})^\omega$ for $\boldsymbol{y} \in \mathbf{T}^m(\mathbf{C})$ and $\omega \in \mathbf{Z}^n$, we conclude that $A(-) \colon \mathbf{T}^m(\mathbf{C}) \to \mathbf{T}^n(\mathbf{C})$ maps $\mathrm{U}(\mathcal{C}A)$ to a subset of $\mathrm{U}(\mathcal{C})$. Now apply Theorem 3.4. ♦

3.3 Rational functions from cones and polytopes

Let $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$ be a half-open rational cone with closure $\mathcal{C} = \overline{\mathcal{C}_0}$ and let $\mathcal{P}_1, \dots, \mathcal{P}_m \subset \mathbf{R}^n_{\geqslant 0}$ be non-empty lattice polytopes. Let $\xi_0, \dots, \xi_m$ be algebraically independent over $\mathbf{Q}$. We now construct a rational function $\mathcal{Z}^{\mathcal{C}_0, \mathcal{P}_1, \dots, \mathcal{P}_m}(\xi_0, \dots, \xi_m)$, which, as we will see in Proposition 3.9, essentially specialises to (3.1).

For $\mathcal{C}_0 = \varnothing$, define $\mathcal{Z}^{\mathcal{C}_0,\mathcal{P}_1,\dots,\mathcal{P}_m}(\xi_0,\dots,\xi_m) := 0$. Henceforth, let $\mathcal{C}_0 \neq \varnothing$. Define $\mathcal{P} := \mathcal{P}_1 + \cdots + \mathcal{P}_m$ to be the Minkowski sum of $\mathcal{P}_1,\dots,\mathcal{P}_m$. It is well-known that the normal fan of $\mathcal{P}$ is precisely the coarsest common refinement of the normal fans of $\mathcal{P}_1,\dots,\mathcal{P}_m$ (see [56, Prop. 7.12]). Specifically, each face τ of $\mathcal{P}$ admits a unique decomposition $\tau = \tau_1 + \cdots + \tau_m$ for suitable faces $\tau_j \subset \mathcal{P}_j$ (see e.g. [25, Prop. 2.1]) and

$$\mathrm{N}_\tau(\mathcal{P}) = \mathrm{N}_{\tau_1}(\mathcal{P}_1) \cap \cdots \cap \mathrm{N}_{\tau_m}(\mathcal{P}_m). \tag{3.2}$$

Write $\mathbf{1} = (1,\dots,1)$. For a vertex $\boldsymbol{v}$ of τ, decomposed as $\boldsymbol{v} = \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_m$ for vertices $\boldsymbol{v}_j$ of τ_j, let $A(\boldsymbol{v}) = [\mathbf{1}^\top, \boldsymbol{v}_1^\top, \dots, \boldsymbol{v}_m^\top] \in \mathrm{M}_{n\times(m+1)}(\mathbf{Z})$. If $\omega \in \mathrm{N}_\tau(\mathcal{P})$, then (3.2) shows that $\omega A(\boldsymbol{v}) = \omega A(\boldsymbol{v}')$ for all vertices $\boldsymbol{v}, \boldsymbol{v}' \in \tau$; by continuity, this identity extends to the closure of $\mathrm{N}_\tau(\mathcal{P})$. Recall the notation for monomial substitutions from §3.2. In view of the remarks following Theorem 3.4, and since $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$, we obtain a well-defined rational function $\mathcal{Z}^{\mathcal{C}_0,\mathcal{P}_1,\dots,\mathcal{P}_m}_\tau(\xi_0,\dots,\xi_m) := |\mathcal{C}_0 \cap \mathrm{N}_\tau(\mathcal{P})|^{A(\boldsymbol{v})}(\xi_0,\dots,\xi_m)$ which does not depend on the choice of a vertex $\boldsymbol{v}$ of τ.

Definition 3.6. $\mathcal{Z}^{\mathcal{C}_0,\mathcal{P}_1,\dots,\mathcal{P}_m}(\xi_0,\dots,\xi_m) := \sum_\tau \mathcal{Z}^{\mathcal{C}_0,\mathcal{P}_1,\dots,\mathcal{P}_m}_\tau(\xi_0,\dots,\xi_m)$, the sum being taken over the $\mathcal{C}_0$-visible faces τ of $\mathcal{P}$ (see Definition 3.1).

Remark 3.7. By construction, $\mathcal{Z}^{\mathcal{C}_0,\mathcal{P}_1,\dots,\mathcal{P}_m}(\xi_0,\dots,\xi_m)$ can be written over a denominator of the form $\prod_{i=1}^r (1 - \xi_0^{a_{i0}} \cdots \xi_m^{a_{im}})$ for $a_{ij} \in \mathbf{N}_0$ and $a_{i0} > 0$.

Proposition 3.8. *Let*

$$U = \{\boldsymbol{z} = (z_0,\dots,z_m) \in \mathbf{C}^{m+1} : 0 < |z_0| < 1 \text{ and } 0 < |z_1|,\dots,|z_m| \leqslant 1\}.$$

Then, for $\boldsymbol{z} \in U$,

$$\mathcal{Z}^{\mathcal{C}_0,\mathcal{P}_1,\dots,\mathcal{P}_m}(\boldsymbol{z}) = \sum_{\omega \in \mathcal{C}_0 \cap \mathbf{Z}^n} z_0^{\langle \mathbf{1},\omega\rangle} \prod_{j=1}^m z_j^{\min(\langle \alpha,\omega\rangle : \alpha \in \mathcal{P}_j)}, \tag{3.3}$$

and the series on the right-hand side converges absolutely and compactly on U.

Proof. Fix a $\mathcal{C}_0$-visible face τ of $\mathcal{P}_1 + \cdots + \mathcal{P}_m$ and a vertex $\boldsymbol{v} = \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_m$ of τ. Let $\mathcal{C}_0^\tau := \mathcal{C}_0 \cap \mathrm{N}_\tau(\mathcal{P})$. Since $\overline{\mathcal{C}_0^\tau} A(\boldsymbol{v})$ is contained in the cone, $\mathcal{C}'$ say, spanned by the rows of $A(\boldsymbol{v})$, we have $\mathrm{U}(\overline{\mathcal{C}_0^\tau} A(\boldsymbol{v})) \supset \mathrm{U}(\mathcal{C}')$ (see Definition 3.3). Moreover, $\mathrm{U}(\mathcal{C}') \supset U$ since $\boldsymbol{v}_j \in \mathbf{R}^n_{\geqslant 0}$ for $j = 1,\dots,m$. By Lemma 3.5, $\mathcal{Z}^{\mathcal{C}_0,\mathcal{P}_1,\dots,\mathcal{P}_m}_\tau(\boldsymbol{z}) = \sum_{\omega \in \mathcal{C}_0^\tau \cap \mathbf{Z}^n} z_0^{\langle \mathbf{1},\omega\rangle} \prod_{j=1}^m z_j^{\langle \boldsymbol{v}_j,\omega\rangle}$ for $\boldsymbol{z} \in U$, the convergence being as stated. The claim follows since $\mathcal{C}_0 = \bigcup_\tau \mathcal{C}_0^\tau$ (disjoint) and $\langle \boldsymbol{v}_j,\omega\rangle = \min(\langle \alpha,\omega\rangle : \alpha \in \mathcal{P}_j)$ for $\omega \in \mathcal{C}_0^\tau$. ♦

3.4 Computing $\mathfrak{P}$-adic integrals associated with cones and polytopes

Let K be a p-adic field with associated objects as in Notation 2.5.

Proposition 3.9. *Let $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$ be a half-open rational cone and let $\mathcal{P}_1,\dots,\mathcal{P}_m \subset \mathbf{R}^n_{\geqslant 0}$ be non-empty lattice polytopes. Let $\mathcal{Z}^{\mathcal{C}_0,\mathcal{P}_1,\dots,\mathcal{P}_m}(\xi_0,\dots,\xi_m)$ be as in Definition 3.6. Then*

$$\int\limits_{\{\boldsymbol{x} \in K^n : \nu(\boldsymbol{x}) \in \mathcal{C}_0\}} \prod_{j=1}^m \|\mathcal{P}_j(\boldsymbol{x})\|^{s_j} \,\mathrm{d}\mu(\boldsymbol{x}) = (1-q^{-1})^n \cdot \mathcal{Z}^{\mathcal{C}_0,\mathcal{P}_1,\dots,\mathcal{P}_m}(q^{-1}, q^{-s_1},\dots,q^{-s_m}) \tag{3.4}$$

for $s_1, \dots, s_m \in \mathbf{C}$ *with* $\mathrm{Re}(s_j) \geqslant 0$.

Proof. For $\omega \in \mathbf{Z}^n$, write $\mathbf{T}^n_\omega(K) := \{\boldsymbol{x} \in \mathbf{T}^n(K) : \nu(\boldsymbol{x}) = \omega\}$. If $\boldsymbol{x} \in \mathbf{T}^n_\omega(K)$, then $\|\mathcal{P}_j(\boldsymbol{x})\| = q^{-\min\left(\langle\alpha,\omega\rangle:\alpha\in\mathcal{P}_j\right)}$. Note that $F_\omega \colon \mathbf{T}^n(\mathfrak{O}) \xrightarrow{\approx} \mathbf{T}^n_\omega(K), \boldsymbol{u} \mapsto \pi^\omega \boldsymbol{u} := (\pi^{\omega_1}u_1, \dots, \pi^{\omega_n}u_n)$ satisfies $|\det(F'_\omega(\boldsymbol{u}))| = q^{-\langle \mathbf{1},\omega\rangle}$ for all $\boldsymbol{u} \in \mathbf{T}^n(\mathfrak{O})$. Since $\mu(\mathbf{T}^n(\mathfrak{O})) = (1-q^{-1})^n$, using the $\mathfrak{P}$-adic change of variables formula [29, Prop. 7.4.1], we obtain

$$\int\limits_{\{\boldsymbol{x}\in K^n:\nu(\boldsymbol{x})\in\mathcal{C}_0\}} \prod_{j=1}^m \|\mathcal{P}_j(\boldsymbol{x})\|^{s_j}\,\mathrm{d}\mu(\boldsymbol{x}) = (1-q^{-1})^n \sum_{\omega\in\mathcal{C}_0\cap\mathbf{Z}^n} q^{-\langle\mathbf{1},\omega\rangle - \sum_{j=1}^m s_j \min\left(\langle\alpha,\omega\rangle:\alpha\in\mathcal{P}_j\right)}$$

whence the claim follows from Proposition 3.8. ♦

Remark 3.10. Note that Proposition 3.9 behaves well under base change. Namely, replacing K by a finite extension $K' \supset K$ simply amounts to replacing q by q^f on the right-hand side of (3.4), where f is the residue class degree of K'/K.

4 Non-degeneracy I: computing $\mathfrak{P}$-adic integrals

In this section, we study a class of $\mathfrak{P}$-adic integrals which includes Igusa-type zeta functions associated with polynomial mappings (see [51]) as well as the integrals in Theorem 2.6. In Theorem 4.10, we give an explicit formula for the integrals considered under suitable non-degeneracy assumptions. In addition to being applicable to the study of zeta functions of groups, algebras, and modules, our findings generalise various existing applications of non-degeneracy in the context of Igusa's local zeta function.

4.1 Newton polytopes and initial forms

The following is mostly folklore. Let R be a commutative ring and let $\boldsymbol{X} = (X_1, \dots, X_n)$ be indeterminates over R. Let $f = \sum_{\alpha\in\mathbf{Z}^n} c_\alpha \boldsymbol{X}^\alpha \in R[\boldsymbol{X}^{\pm1}]$, where $c_\alpha \in R$, almost all of which are zero. The **support** of f is $\mathrm{supp}(f) := \{\alpha \in \mathbf{Z}^n : c_\alpha \neq 0\}$. The **Newton polytope** $\mathrm{New}(f)$ of f is the convex hull of $\mathrm{supp}(f)$ within $\mathbf{R}^n$. If R is a domain, then $\mathrm{New}(fg) = \mathrm{New}(f) + \mathrm{New}(g)$ for $f, g \in R[\boldsymbol{X}^{\pm1}]$ (see [48, Lemma 2.2]). For $\omega \in \mathbf{R}^n$, define the **initial form** $\mathrm{in}_\omega(f)$ of f in the direction ω to be the sum of all those monomials $c_\alpha \boldsymbol{X}^\alpha$ (with $\alpha \in \mathrm{supp}(f)$) where $\langle\alpha,\omega\rangle$ attains its minimum. Let $f \neq 0$. We then have $\mathrm{face}_\omega(\mathrm{New}(f)) = \mathrm{New}(\mathrm{in}_\omega(f))$ for $\omega \in \mathbf{R}^n$ [48, (2.5)]. The equivalence classes of $\sim$ defined on $\mathbf{R}^n$ via $\omega \sim \omega'$ if and only if $\mathrm{in}_\omega(f) = \mathrm{in}_{\omega'}(f)$ are precisely the normal cones of the faces of $\mathrm{New}(f)$ as defined in §3.1.

Let Y be another variable over R. For $\omega \in \mathbf{Z}^n$, we write $Y^\omega \boldsymbol{X} = (Y^{\omega_1}X_1, \dots, Y^{\omega_n}X_n)$. Let $f \in R[\boldsymbol{X}^{\pm1}]$ be non-zero and let τ be a face of $\mathrm{New}(f)$. Write $f = \sum_{\alpha\in\mathbf{Z}^n} c_\alpha \boldsymbol{X}^\alpha$ as above and let $\omega \in \mathrm{N}_\tau(\mathrm{New}(f)) \cap \mathbf{Z}^n$. Choose a vertex $\Lambda(\tau)$ of τ. We may write

$$f(Y^\omega\boldsymbol{X}) = Y^{\langle\Lambda(\tau),\omega\rangle} \cdot \Bigl(\mathrm{in}_\omega(f) + Y \cdot \Bigl(\sum_{\alpha \text{ s.t. } \langle\alpha-\Lambda(\tau),\omega\rangle>0} Y^{\langle\alpha-\Lambda(\tau),\omega\rangle-1} c_\alpha \boldsymbol{X}^\alpha\Bigr)\Bigr).$$

Note that $\langle\Lambda(\tau),\omega\rangle$ is independent of the choice of $\Lambda(\tau)$.

Notation 4.1. $f_\omega(\boldsymbol{X}, Y) := Y^{-\langle\Lambda(\tau),\omega\rangle} \cdot f(Y^\omega\boldsymbol{X}) \in R[\boldsymbol{X}^{\pm1}, Y]$.

4.2 Non-degeneracy

Let k be a field and let $X_1, \dots, X_n$ be algebraically independent over k. Write $\boldsymbol{X} = (X_1, \dots, X_n)$. Let $\boldsymbol{f} = (f_i)_{i \in I}$ be a finite family of non-zero elements of $k[\boldsymbol{X}^{\pm 1}]$ and let $\mathcal{C}_0 \subset \mathbf{R}^n$ be a half-open rational cone. Let $\mathcal{N} := \mathrm{New}\big(\prod_{i \in I} f_i\big) = \sum_{i \in I} \mathrm{New}(f_i)$. For a face $\tau \subset \mathcal{N}$, let $\mathcal{C}_0^\tau := \mathcal{C}_0 \cap \mathrm{N}_\tau(\mathcal{N})$. By (3.2) and §4.1, we may unambiguously define $f_i^\tau := \mathrm{in}_\omega(f_i)$ for $i \in I$ and an arbitrary $\omega \in \mathrm{N}_\tau(\mathcal{N})$. Let $\bar{k}$ be an algebraic closure of k.

Definition 4.2.

(i) We say that $\boldsymbol{f}$ is **non-degenerate relative to** $\mathcal{C}_0$ if the following holds:

for all $\mathcal{C}_0$-visible faces $\tau \subset \mathcal{N}$ and all $J \subset I$, if $\boldsymbol{u} \in \mathbf{T}^n(\bar{k})$ satisfies $f_j^\tau(\boldsymbol{u}) = 0$ for all $j \in J$, then the Jacobian matrix $\left[\frac{\partial f_j^\tau(\boldsymbol{u})}{\partial X_i}\right]_{i=1,\dots,n; j \in J}$ has rank $\#J$.

(ii) We say that $\boldsymbol{f}$ is **globally non-degenerate** if it is non-degenerate relative to $\mathbf{R}^n$.

Remark 4.3.

(i) Non-degeneracy of $\boldsymbol{f}$ relative to $\mathcal{C}_0$ is preserved under extension of k.

(ii) Let $f \in k[\boldsymbol{X}^{\pm 1}]$ and write $f' = \left(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right)$. Then $(\boldsymbol{X}^\gamma f)' \equiv \boldsymbol{X}^\gamma \cdot f' \bmod f$ for $\gamma \in \mathbf{Z}^n$, the congruence being understood componentwise and within $k[\boldsymbol{X}^{\pm 1}]$. In particular, whether $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$ or not is invariant under rescaling of the elements of $\boldsymbol{f}$ by Laurent monomials.

(iii) Let k be a number field with ring of integers $\mathfrak{o}$. By the (weak) Nullstellensatz, if $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$ over k, then, for almost all primes $\mathfrak{p}$ of k, the reduction of $\boldsymbol{f}$ modulo $\mathfrak{p}$ (that is, the image of $\boldsymbol{f}$ under the natural map $\mathfrak{o}_\mathfrak{p}[\boldsymbol{X}^{\pm 1}] \to (\mathfrak{o}/\mathfrak{p})[\boldsymbol{X}^{\pm 1}]$) is non-degenerate relative to $\mathcal{C}_0$ over $\mathfrak{o}/\mathfrak{p}$.

(iv) If all polynomials in $\boldsymbol{f}$ are homogeneous and $(1, \dots, 1)$ is an interior point of $\mathcal{C}_0$ (which thus needs to be full-dimensional), then every face of $\mathcal{N}$ is $\mathcal{C}_0$-visible whence $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$ if and only if it is globally non-degenerate.

Remark 4.4. For notational convenience, we also consider sets of polynomials instead of families, in particular in Theorems 4.10 and 6.7. When we speak of non-degeneracy of a finite set $\boldsymbol{f} \subset k[\boldsymbol{X}^{\pm 1}]$ (with $0 \notin \boldsymbol{f}$), we refer to properties of the family $(f)_{f \in \boldsymbol{f}}$.

Khovanskii [33, 34], Kushnirenko [37], Varchenko [49], and others investigated complex varieties defined by suitably non-degenerate systems of polynomials (see e.g. Theorem 6.3). We may rephrase Khovanskii's notion of non-degeneracy [33, §2] as follows.

Definition 4.5. We say $\boldsymbol{f} = (f_i)_{i \in I}$ is **$\cap$-non-degenerate** if the following holds: for every face $\tau \subset \mathcal{N}$ and $\boldsymbol{u} \in \mathbf{T}^n(\bar{k})$, if $f_i^\tau(\boldsymbol{u}) = 0$ for all $i \in I$, then the Jacobian matrix $\left[\frac{\partial f_j^\tau(\boldsymbol{u})}{\partial X_i}\right]_{i=1,\dots,n; j \in I}$ has rank $\#I$.

Hence, using the conventions from Remark 4.4, a finite set $\boldsymbol{f} \subset k[\boldsymbol{X}^{\pm 1}]$ (with $0 \notin \boldsymbol{f}$) is globally non-degenerate in our sense if and only if each subset $\boldsymbol{g} \subset \boldsymbol{f}$ is $\cap$-non-degenerate.

4.3 Computing non-degenerate $\mathfrak{P}$-adic integrals

Let k be a number field with ring of integers $\mathfrak{o}$. As before, write $\boldsymbol{X} = (X_1, \dots, X_n)$. Let $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$ be a half-open rational cone and let $\boldsymbol{f}_0 \subset k[\boldsymbol{X}^{\pm 1}]$ and $\boldsymbol{f}_1, \dots, \boldsymbol{f}_m \subset k[\boldsymbol{X}]$ be non-empty finite sets of non-zero (Laurent) polynomials.

Definition 4.6. For a p-adic field $K \supset k$ with associated data as in Notation 2.5, let

$$\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m) := \int\limits_{\{\boldsymbol{x} \in \mathbf{T}^n(K) : \nu(\boldsymbol{x}) \in \mathcal{C}_0, \|\boldsymbol{f}_0(\boldsymbol{x})\| \leqslant 1\}} \|\boldsymbol{f}_1(\boldsymbol{x})\|^{s_1} \cdots \|\boldsymbol{f}_m(\boldsymbol{x})\|^{s_m} \,\mathrm{d}\mu(\boldsymbol{x}),$$

where $s_1, \dots, s_m \in \mathbf{C}$ with $\mathrm{Re}(s_j) \geqslant 0$.

In [20], integrals of the form (2.1) were studied within the framework of cone integrals introduced there. As we will explain in Remark 4.12, $\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m)$ specialises to the integral in (2.1). (More generally, every cone integral with monomial left-hand sides of divisibility conditions arises as a specialisation of an integral $\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m)$.) Furthermore, taking $\mathcal{C}_0 = \mathbf{R}^n_{\geqslant 0}$, $m = 1$, and $\boldsymbol{f}_0 = \{1\}$ in Definition 4.6, as $\mu(K^n \setminus \mathbf{T}^n(K)) = 0$, we recover the Igusa-type integral $\int_{\mathfrak{O}^n} \|\boldsymbol{f}_1(\boldsymbol{x})\|^s \,\mathrm{d}\mu(\boldsymbol{x})$ considered in [51].

Under the assumption that the set $\boldsymbol{f} := \boldsymbol{f}_0 \cup \dots \cup \boldsymbol{f}_m$ in Definition 4.6 is non-degenerate relative to $\mathcal{C}_0$, in this subsection, we derive an explicit formula for $\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m)$ (see Theorem 4.10) which is valid for all p-adic fields $K \supset k$ such that the associated prime $\mathfrak{P} \cap \mathfrak{o}$ of k does not belong to some finite exceptional set (depending on $\boldsymbol{f}$ and $\mathcal{C}_0$). Since we are willing to ignore finitely many primes of k, we may assume that $\boldsymbol{f} \subset \mathfrak{o}[\boldsymbol{X}^{\pm 1}]$.

Remark 4.7. Although we will not need this in the sequel, we note that it is possible to produce an "explicit formula" (in Denef's sense [15, §3]) for $\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m)$ in terms of a principalisation of ideals over k, cf. [1, Prop. 4.1] where a similar class of integrals is studied using techniques going back to [51, 53]. In practice, finding a principalisation of ideals is closely related to finding an embedded resolution of singularities (see [54, Prop. 2.5.1]) and thus equally impractical, in general.

Remark 4.8. The role played by the ambient half-open cone $\mathcal{C}_0$ in Definition 4.6 might seem artificial. Namely, if $\mathcal{C}_0$ is closed, then by suitably modifying $\boldsymbol{f}_0$, we may reduce the computation of $\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m)$ to the case $\mathcal{C}_0 = \mathbf{R}^n_{\geqslant 0}$; the case of a general $\mathcal{C}_0$ then follows by the inclusion-exclusion principle. However, the more practically-minded extensions in [44] of the techniques described in the present article rely on successive refinements of the partition $\mathcal{C}_0 = \bigcup_\tau \mathcal{C}_0^\tau$ used here and the author found these to be most conveniently expressed in terms of partitions of $\mathcal{C}_0$ itself.

Setup: associated cones, polytopes, and sets. As in §4.2, let $\mathcal{N} := \mathrm{New}(\prod \boldsymbol{f}) = \sum_{f \in \boldsymbol{f}} \mathrm{New}(f)$ and $f^\tau := \mathrm{in}_\omega(f)$ for $f \in \boldsymbol{f}$, a $\mathcal{C}_0$-visible face $\tau \subset \mathcal{N}$, and an arbitrary $\omega \in \mathcal{C}_0^\tau := \mathcal{C}_0 \cap \mathrm{N}_\tau(\mathcal{N})$. Given τ, let $\tau = \sum_{f \in \boldsymbol{f}} \tau(f)$ be the decomposition of τ into faces $\tau(f) \subset \mathrm{New}(f)$. Given f and τ, we choose, once and for all, a vertex $\Lambda(f, \tau)$ of $\tau(f)$.

Definition 4.9. Let $\tau \subset \mathcal{N}$ be a $\mathcal{C}_0$-visible face (see Definition 3.1) and let $\boldsymbol{g} \subset \boldsymbol{f}$.

(i) For $j = 0, \dots, m$, define a lattice polytope

$$\mathcal{P}_j^\tau(\boldsymbol{g}) := \operatorname{conv}\Big(\Big\{\big(\Lambda(g,\tau), \boldsymbol{e}_g\big) : g \in \boldsymbol{f}_j \cap \boldsymbol{g}\Big\} \cup \Big\{\big(\Lambda(f,\tau), 0\big) : f \in \boldsymbol{f}_j \setminus \boldsymbol{g}\Big\}\Big)$$

within $\mathbf{R}^n \times \mathbf{R}^{\boldsymbol{g}}$, where $(\boldsymbol{e}_g)_{g\in\boldsymbol{g}}$ is the standard basis of $\mathbf{R}^{\boldsymbol{g}} = \{(x_g)_{g\in\boldsymbol{g}} : \forall g \in \boldsymbol{g} \colon x_g \in \mathbf{R}\} \approx \mathbf{R}^{|\boldsymbol{g}|}$.

(ii) Define a half-open cone

$$\mathcal{C}_0^\tau(\boldsymbol{g}) := (\mathcal{C}_0^\tau \times \mathbf{R}_{>0}^{\boldsymbol{g}}) \cap \mathcal{P}_0^\tau(\boldsymbol{g})^*,$$

where $\mathcal{P}_0^\tau(\boldsymbol{g})^*$ denotes the dual cone of $\mathcal{P}_0^\tau(\boldsymbol{g})$ as in §3.1.

(iii) For a prime $\mathfrak{p}$ of k and a field extension $\mathfrak{k}$ of $\mathfrak{o}/\mathfrak{p}$, let

$$\bar{V}_{\boldsymbol{g}}^\tau(\mathfrak{k}) := \{\boldsymbol{u} \in \mathbf{T}^n(\mathfrak{k}) : \forall f \in \boldsymbol{f} \colon f^\tau(\boldsymbol{u}) = 0 \iff f \in \boldsymbol{g}\}.$$

Theorem 4.10. *Let k be a number field with ring of integers $\mathfrak{o}$. Let $\mathcal{C}_0 \subset \mathbf{R}_{\geqslant 0}^n$ be a half-open rational cone and let $\boldsymbol{f}_0 \subset \mathfrak{o}[\boldsymbol{X}^{\pm 1}]$ and $\boldsymbol{f}_1, \dots, \boldsymbol{f}_m \subset \mathfrak{o}[\boldsymbol{X}] = \mathfrak{o}[X_1, \dots, X_n]$ be non-empty finite sets with $0 \notin \boldsymbol{f} := \boldsymbol{f}_0 \cup \dots \cup \boldsymbol{f}_m$. Define local zeta functions $\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m)$ as in Definition 4.6. For a $\mathcal{C}_0$-visible face τ of $\mathcal{N} := \operatorname{New}(\prod \boldsymbol{f})$ (see Definition 3.1 and §4.1) and a subset $\boldsymbol{g} \subset \boldsymbol{f}$, define a half-open cone $\mathcal{C}_0^\tau(\boldsymbol{g})$, a lattice polytope $\mathcal{P}_j^\tau(\boldsymbol{g})$, and finite sets $\bar{V}_{\boldsymbol{g}}^\tau(\mathfrak{k})$ as in Definition 4.9(i)–(iii). Finally, define $\mathcal{Z}^{\mathcal{C}_0^\tau(\boldsymbol{g}), \mathcal{P}_1^\tau(\boldsymbol{g}), \dots, \mathcal{P}_m^\tau(\boldsymbol{g})}(\xi_0, \dots, \xi_m) \in \mathbf{Q}(\xi_0, \dots, \xi_m)$ as in Definition 3.6.*

Suppose that $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$ over k in the sense of Definition 4.2(i). Then there exists a finite set S of primes of k such that the following holds: if K is a non-Archimedean local field extending k such that the associated prime ideal $\mathfrak{P}$ of its valuation ring $\mathfrak{O}$ satisfies $\mathfrak{P} \cap \mathfrak{o} \notin S$, then, writing $q = |\mathfrak{O}/\mathfrak{P}|$, we have

$$\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m) = \sum_{\substack{\boldsymbol{g} \subset \boldsymbol{f}, \\ \tau \subset \mathcal{N}}} \#\bar{V}_{\boldsymbol{g}}^\tau(\mathfrak{O}/\mathfrak{P}) \cdot \frac{(q-1)^{|\boldsymbol{g}|}}{q^n} \cdot \mathcal{Z}^{\mathcal{C}_0^\tau(\boldsymbol{g}), \mathcal{P}_1^\tau(\boldsymbol{g}), \dots, \mathcal{P}_m^\tau(\boldsymbol{g})}(q^{-1}, q^{-s_1}, \dots, q^{-s_m})$$

for all $s_1, \dots, s_m \in \mathbf{C}$ with $\operatorname{Re}(s_j) \geqslant 0$ and τ ranging over the $\mathcal{C}_0$-visible faces of $\mathcal{N}$.

Remark. Following [16], the established method of computing $\mathfrak{P}$-adic integrals associated with non-degenerate polynomials in the literature (see §4.4) is to express them as countable sums of integrals over copies of $\mathbf{T}^n(\mathfrak{O})$, each of which can then computed using Hensel's lemma. Our proof of Theorem 4.10 proceeds along the same lines.

Proof of Theorem 4.10.

(i) Restrictions on the local field:

Choose $\gamma \in \mathbf{Z}^n$ such that $\tilde{f} := \boldsymbol{X}^\gamma f$ belongs to $\mathfrak{o}[\boldsymbol{X}]$ for all $f \in \boldsymbol{f}$. By Remark 4.3(ii), $\tilde{\boldsymbol{f}} := \big(\tilde{f} : f \in \boldsymbol{f}\big)$ is non-degenerate relative to $\mathcal{C}_0$. According to Remark 4.3(iii), for almost all primes $\mathfrak{p}$ of k, non-degeneracy of $\tilde{\boldsymbol{f}}$ relative to $\mathcal{C}_0$ is preserved under reduction modulo $\mathfrak{p}$. We assume that $\mathfrak{P} \cap \mathfrak{o}$ is among these primes. Finally, we assume that the non-zero coefficients of each $f \in \boldsymbol{f}$ are $\mathfrak{P}$-adic units.

(ii) Breaking up the integral:

Write $T(\bar{\boldsymbol{u}})$ for the fibre of $\bar{\boldsymbol{u}} \in \mathbf{T}^n(\mathfrak{O}/\mathfrak{P})$ under the natural map $\mathbf{T}^n(\mathfrak{O}) \to \mathbf{T}^n(\mathfrak{O}/\mathfrak{P})$. By decomposing $\mathbf{T}^n(K)$ into subsets $\mathbf{T}^n_\omega(K) \approx \mathbf{T}^n(\mathfrak{O})$ as in the proof of Proposition 3.9 and by further decomposing $\mathbf{T}^n(\mathfrak{O}) = \bigcup_{\bar{u}\in\mathbf{T}^n(\mathfrak{O}/\mathfrak{P})} T(\bar{u})$, we obtain

$$\mathsf{Z}_K^{\mathcal{C}_0,\boldsymbol{f}_0,\dots,\boldsymbol{f}_m}(s_1,\dots,s_m) = \sum_{\tau}\sum_{\boldsymbol{g}\subset\boldsymbol{f}}\sum_{\bar{\boldsymbol{u}}\in\bar{V}^\tau_{\boldsymbol{g}}(\mathfrak{O}/\mathfrak{P})}\sum_{\omega\in\mathcal{C}_0^\tau\cap\mathbf{Z}^n} q^{-\langle\mathbf{1},\omega\rangle} I_\omega(\bar{\boldsymbol{u}}), \tag{4.1}$$

where $I_\omega(\bar{\boldsymbol{u}}) := \int_{\{\boldsymbol{w}\in T(\bar{\boldsymbol{u}}):\|\boldsymbol{f}_0(\pi^\omega\boldsymbol{w})\|\leqslant 1\}} \prod_{j=1}^m \|\boldsymbol{f}_j(\pi^\omega\boldsymbol{w})\|^{s_j}\,\mathrm{d}\mu(\boldsymbol{w})$.

(iii) Computing the pieces:

Write f_ω and $\tilde{f}_\omega$ for $f_\omega(\boldsymbol{X},\pi)$ and $\tilde{f}_\omega(\boldsymbol{X},\pi)$ (see Notation 4.1), respectively; note that $\mathrm{in}_\omega(\tilde{f}) = \boldsymbol{X}^\gamma\,\mathrm{in}_\omega(f)$ and $\tilde{f}_\omega = \boldsymbol{X}^\gamma f_\omega$. For $\omega \in \mathcal{C}_0^\tau\cap\mathbf{Z}^n$, $\boldsymbol{u}\in\mathbf{T}^n(\mathfrak{O})$, and $f\in\boldsymbol{f}$, we have $f(\pi^\omega\boldsymbol{u}) = \pi^{\langle\Lambda(f,\tau),\omega\rangle}\boldsymbol{u}^{-\gamma}\tilde{f}_\omega(\boldsymbol{u})$ and $\tilde{f}_\omega \equiv \mathrm{in}_\omega(\tilde{f}) \bmod \mathfrak{P}$. Fix $\tau\subset\mathcal{N}$, $\omega\in\mathcal{C}_0^\tau\cap\mathbf{Z}^n$, $\boldsymbol{g}\subset\boldsymbol{f}$, and $\bar{\boldsymbol{u}}\in\bar{V}^\tau_{\boldsymbol{g}}(\mathfrak{O}/\mathfrak{P})$. Let $e = |\boldsymbol{g}|$, $r = |\boldsymbol{f}|$, $\boldsymbol{f} = \{f_1,\dots,f_r\}$, and $\boldsymbol{g} = \{f_1,\dots,f_e\}$. Let $\boldsymbol{u} = (u_1,\dots,u_n)\in\mathbf{T}^n(\mathfrak{O})$ be an arbitrary lift of $\bar{\boldsymbol{u}}$.

By non-degeneracy of $\tilde{\boldsymbol{f}}$ over $\mathfrak{O}/\mathfrak{P}$, the Jacobian matrix of $(\tilde{f}_{1,\omega},\dots,\tilde{f}_{e,\omega})$ at $\boldsymbol{u}$ has rank e modulo $\mathfrak{P}$. In particular, $e\leqslant n$ and after suitably renumbering variables, we may thus assume that $\det\boldsymbol{h}'(\boldsymbol{u})\not\equiv 0 \bmod \mathfrak{P}$, where $\boldsymbol{h} := (\tilde{f}_{1,\omega},\dots,\tilde{f}_{e,\omega},X_{e+1}-u_{e+1},\dots,X_n-u_n)$. Hensel's lemma for polynomial mappings [7, Ch. III, §4.5, Cor. 2] allows us to replace $u_1,\dots,u_e$ by elements in the same residue classes modulo $\mathfrak{P}$ in such a way that $\boldsymbol{h}(\boldsymbol{u}) = 0$ holds. The $\mathfrak{P}$-adic inverse function theorem [7, Ch. III, §4.5, Prop. 7] (applied to $\boldsymbol{h}(\boldsymbol{u}+\boldsymbol{X})$) then shows that $\boldsymbol{h}$ induces a bianalytic and measure-preserving bijection from $T(\bar{\boldsymbol{u}}) = \boldsymbol{u}+\mathfrak{P}^n$ onto $\mathfrak{P}^n$. For $\boldsymbol{y} = (y_1,\dots,y_n)\in\mathfrak{P}^n$, $\boldsymbol{v} := \boldsymbol{h}^{-1}(\boldsymbol{y})-\boldsymbol{u}\in\mathfrak{P}^n$, and $1\leqslant i\leqslant r$, we then have $|f_i(\pi^\omega(\boldsymbol{u}+\boldsymbol{v}))| = |\pi^{\langle\Lambda(f_i,\tau),\omega\rangle}\tilde{f}_{i,\omega}(\boldsymbol{u}+\boldsymbol{v})|$ with $|\tilde{f}_{i,\omega}(\boldsymbol{u}+\boldsymbol{v})| = 1$ for $e+1\leqslant i\leqslant r$ and $\tilde{f}_{i,\omega}(\boldsymbol{u}+\boldsymbol{v}) = \tilde{f}_{i,\omega}(\boldsymbol{h}^{-1}(\boldsymbol{y})) = y_i$ for $1\leqslant i\leqslant e$. For $0\leqslant j\leqslant m$, define $\varepsilon_{ij} = 1$ if $f_i\in\boldsymbol{f}_j$ and $\varepsilon_{ij} = 0$ otherwise. We then have

$$\begin{aligned}\|\boldsymbol{f}_j(\pi^\omega(\boldsymbol{u}+\boldsymbol{v}))\| = \Big\|&\varepsilon_{1j}\pi^{\langle\Lambda(f_1,\tau),\omega\rangle}y_1,\dots,\varepsilon_{ej}\pi^{\langle\Lambda(f_e,\tau),\omega\rangle}y_e,\\ &\varepsilon_{e+1,j}\pi^{\langle\Lambda(f_{e+1},\tau),\omega\rangle},\dots,\varepsilon_{rj}\pi^{\langle\Lambda(f_r,\tau),\omega\rangle}\Big\| =: a_{j\omega}(y_1,\dots,y_e).\end{aligned}$$

Hence, a change of variables yields

$$\begin{aligned}I_\omega(\bar{\boldsymbol{u}}) &= \mu_{n-e}(\mathfrak{P}^{n-e})\cdot\int\limits_{\{\boldsymbol{y}\in\mathfrak{P}^e:a_{0\omega}(\boldsymbol{y})\leqslant 1\}} a_{1\omega}(\boldsymbol{y})^{s_1}\cdots a_{m\omega}(\boldsymbol{y})^{s_m}\,\mathrm{d}\mu_e(\boldsymbol{y})\\ &= \frac{(1-q^{-1})^e}{q^{n-e}}\sum_{\substack{\psi\in\mathbf{N}^e,\\ a_{0\omega}(\pi^\psi)\leqslant 1}} q^{-\langle\mathbf{1},\psi\rangle}a_{1\omega}(\pi^\psi)^{s_1}\cdots a_{m\omega}(\pi^\psi)^{s_m}\\ &= \frac{(q-1)^e}{q^n}\sum_{\substack{\psi\in\mathbf{N}^e,\\ \varepsilon_{i0}\cdot(\langle\Lambda(f_i,\tau),\omega\rangle+\psi_i)\geqslant 0\ (1\leqslant i\leqslant e),\\ \varepsilon_{i0}\cdot\langle\Lambda(f_i,\tau),\omega\rangle\geqslant 0\ (e+1\leqslant i\leqslant r)}} q^{-\langle\mathbf{1},\psi\rangle}\prod_{j=1}^m q^{-s_j\min\begin{pmatrix}\varepsilon_{ij}\cdot(\langle\Lambda(f_i,\tau),\omega\rangle+\psi_i)\ (1\leqslant i\leqslant e),\\ \varepsilon_{ij}\cdot\langle\Lambda(f_i,\tau),\omega\rangle\ (e+1\leqslant i\leqslant r)\end{pmatrix}},\end{aligned}$$

where we wrote μ_d for the normalised Haar measure on $\mathfrak{O}^d$ and $\pi^\psi = (\pi^{\psi_1}, \dots, \pi^{\psi_e})$.

Using Proposition 3.8, the claim now follows from (4.1). ♦

Example 4.11 (Igusa's local zeta function)**.** Let $f \in k[\boldsymbol{X}]$ be non-constant. Denef and Hoornaert [16, §4] gave an explicit formula for the Igusa zeta function $\int_{\mathfrak{O}^n} |f(\boldsymbol{x})|^s \,\mathrm{d}\mu(\boldsymbol{x})$ (for $k = \mathbf{Q}$, $K = \mathbf{Q}_p$) if (f) is non-degenerate relative to $\mathbf{R}^n_{\geqslant 0}$ in our terminology. As we will explain in §4.4, Theorem 4.10 essentially generalises their result. In fact, even in the very special case of a single polynomial, our method applies to a larger class of polynomials. Namely, let $f = f_1^{e_1} \cdots f_m^{e_m}$ be the factorisation into powers of distinct irreducibles and suppose that $(f_1, \dots, f_m)$ is non-degenerate relative to $\mathbf{R}^n_{\geqslant 0}$. By applying Theorem 4.10 with $\boldsymbol{f}_0 = \{1\}$, $\boldsymbol{f}_1 = \{f_1\}, \dots, \boldsymbol{f}_m = \{f_m\}$, we may compute $\int_{\mathfrak{O}^n} |f_1(\boldsymbol{x})|^{s_1} \cdots |f_m(\boldsymbol{x})|^{s_m} \,\mathrm{d}\mu(\boldsymbol{x})$ and hence $\int_{\mathfrak{O}^n} |f(\boldsymbol{x})|^s \,\mathrm{d}\mu(\boldsymbol{x})$ via $s_j = e_j s$ for $j = 1, \dots, m$. Non-degeneracy of (f) implies that of $(f_1, \dots, f_m)$ but the converse is false.

Remark 4.12. There are various ways of interpreting the integral in Theorem 2.6 within the framework of Theorem 4.10. For instance, we can regard the integrand as a specialisation of $|x_{11}|^{s_1} \cdots |x_{dd}|^{s_d}$. On the level of the rational functions in Theorem 4.10, such specialisations amount to monomial substitutions as in §3.2. Specifically, let γ and δ be independent variables over $\mathbf{Q}$. Remark 3.7 shows that we may define $\tilde{\mathcal{Z}}_{\boldsymbol{g}}^\tau(\gamma, \delta) := \mathcal{Z}^{\mathcal{C}_0^\tau(\boldsymbol{g}), \mathcal{P}_1^\tau(\boldsymbol{g}), \dots, \mathcal{P}_m^\tau(\boldsymbol{g})}(\gamma, \gamma^{-1}\delta, \dots, \gamma^{-m}\delta) \in \mathbf{Q}(\gamma, \delta)$ for $\boldsymbol{g} \subset \boldsymbol{f}$ and $\tau \subset \mathcal{N}$. Hence, under the same assumptions as in Theorem 4.10, for $\mathrm{Re}(s) \geqslant m$, we have

$$\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s-1, \dots, s-m) = \sum_{\substack{\boldsymbol{g} \subset \boldsymbol{f}, \\ \tau \subset \mathcal{N}}} \#\bar{V}_{\boldsymbol{g}}^\tau(\mathfrak{O}/\mathfrak{P}) \cdot \frac{(q-1)^{|\boldsymbol{g}|}}{q^n} \cdot \tilde{\mathcal{Z}}_{\boldsymbol{g}}^\tau(q^{-1}, q^{-s}).$$

The central feature of Theorem 4.10 is that it is entirely explicit. In contrast, the general formulae for subalgebra zeta functions given in [20, 53] rely on the usually infeasible step of constructing a resolution of singularities. As an illustration of the key ingredients of Theorem 4.10 and as a demonstration of its practical value, in §7.1, we use it to compute the known $\mathfrak{P}$-adic subalgebra zeta functions of $\mathfrak{sl}_2(\mathbf{Z})$ for $\mathfrak{P} \cap \mathbf{Z} \neq \langle 2 \rangle$.

4.4 Non-degeneracy and $\mathfrak{P}$-adic integration in the literature

Let k be a field with algebraic closure $\bar{k}$. In applications to Igusa-type zeta functions (see e.g. [16, 51]), non-degeneracy of a non-zero polynomial $f \in k[\boldsymbol{X}]$ is usually expressed not in terms of the Newton polytope $\mathcal{N} := \mathrm{New}(f)$ but in terms of the Newton polyhedron $\mathcal{P} := \mathcal{N} + \mathbf{R}^n_{\geqslant 0}$. This approach is subsumed by ours as follows. The normal fan of $\mathcal{P}$ is the least common refinement of the normal fan of $\mathcal{N}$ and the fan of faces of $\mathbf{R}^n_{\geqslant 0}$. Hence, the partition $\mathbf{R}^n_{\geqslant 0} = \bigcup_\gamma \mathrm{N}_\gamma(\mathcal{P})$ into (relatively open) normal cones of non-empty faces $\gamma \subset \mathcal{P}$ refines our partition $\mathbf{R}^n_{\geqslant 0} = \bigcup_\tau (\mathrm{N}_\tau(\mathcal{N}) \cap \mathbf{R}^n_{\geqslant 0})$ indexed by $\mathbf{R}^n_{\geqslant 0}$-visible faces $\tau \subset \mathcal{N}$. The number of faces of $\mathcal{P}$ will often greatly exceed that of $\mathcal{N}$ which leads to considerable practical limitations of the former approach. We note that by Lemma 3.2, the compact faces of $\mathcal{P}$ often considered in the literature coincide with the $\mathbf{R}^n_{>0}$-visible faces of $\mathcal{N}$ in our terminology. In this sense, in addition to being better suited for explicit computations, Theorem 4.10 provides a far-reaching generalisation of [6, Thm 4.3] (itself a generalisation of [16, Thm 4.2]) on integrals $\int_{\mathbf{Z}_p^n} |f(\boldsymbol{x})|^s |g(\boldsymbol{x})| \,\mathrm{d}\mu(\boldsymbol{x})$; the assumptions used in [6] are equivalent to (f, g) being non-degenerate relative to $\mathbf{R}^n_{\geqslant 0}$ in our sense.

Veys and Zúñiga-Galindo [51] used yet another notion of non-degeneracy for $\boldsymbol{f} = (f_i)_{i\in I}$ to compute integrals $\int_{\mathfrak{O}^n} \|\boldsymbol{f}(\boldsymbol{x})\|^s \,\mathrm{d}\mu(\boldsymbol{x})$. Their notion is based on the polyhedron $\mathrm{conv}\big(\bigcup_{i\in I} \mathrm{New}(f_i)\big) + \mathbf{R}^n_{\geqslant 0}$. The aforementioned computational difficulties surrounding unbounded polyhedra apply here too. More importantly, we found the non-degeneracy assumption in [51, Def. 3.1] to be too restrictive for our intended applications to subalgebra and submodule zeta functions. The Newton polytope $\mathcal{N} = \mathrm{New}(\prod_i f_i)$ central to our approach previously featured in work of Zúñiga-Galindo [57]; it frequently appears in applications of non-degeneracy to problems in Archimedean settings, see e.g. [43].

5 Topological zeta functions

The topological zeta function $\mathsf{Z}_{f,\mathrm{top}}(\boldsymbol{s})$ associated with a polynomial $f \in k[\boldsymbol{X}]$ over a number field, say, is usually defined in terms of numerical data extracted from an embedded resolution of singularities of $\mathrm{Spec}(k[\boldsymbol{X}]/f) \subset \mathrm{Spec}(k[\boldsymbol{X}])$. There are two established ways, both of them due to Denef and Loeser, of proving independence of the chosen resolution (for a third approach, see [41, §1.5]). In [17], Denef and Loeser used ℓ-adic interpolation techniques to express $\mathsf{Z}_{f,\mathrm{top}}(\boldsymbol{s})$ as a limit of Igusa zeta functions associated with f, thus giving meaning to the limit "$p \to 1$" from the introduction. A more geometric approach is given in [18], where the motivic zeta function associated with f is introduced and shown to specialise to $\mathsf{Z}_{f,\mathrm{top}}(\boldsymbol{s})$.

In this section, based on ideas from [17], we give a self-contained introduction to topological zeta functions using the original ℓ-adic approach. We attach topological zeta functions to systems of local zeta functions of a certain shape arising in $\mathfrak{P}$-adic integration. Our approach therefore applies to a range of algebraic counting problems. In particular, in §5.4, we give rigorous definitions of the topological subalgebra and submodule zeta functions informally introduced in §1. The first of these zeta functions was defined in [21, §8] (in greater generality and slightly differently, see Remark 5.18) as a specialisation of a suitable motivic one. In contrast to the more recent theory of motivic integration, $\mathfrak{P}$-adic integration is firmly established as a tool in asymptotic algebra. By relying exclusively on $\mathfrak{P}$-adic methods, the machinery in this section can, for instance, immediately be used to define topological representation zeta functions associated with finitely generated torsion-free nilpotent groups via [53, Prop. 3.1].

5.1 Formal and ℓ-adic binomial expansions

In this subsection, which formalises ideas used in [17, (2.4)], we explain how certain rational functions $W(\boldsymbol{q}, \boldsymbol{t})$ admit an algebraic "reduction modulo $\boldsymbol{q} - 1$" which corresponds to taking an ℓ-adic limit $x \to 1$ of $W(x, x^{-s})$.

Let k be a field of characteristic zero and $\boldsymbol{z}$ be an indeterminate over k. Given a formal power series $f \in k[\![\boldsymbol{z}]\!]$ without constant term, it is well-known that formal powers defined using the binomial series $(1+f)^a = \sum_{d=0}^{\infty}\binom{a}{d} f^d$ with $a \in k$ and $\binom{a}{d} = \frac{a(a-1)\cdots(a-d+1)}{d!}$ satisfy $(1+f)^a(1+f)^b = (1+f)^{a+b}$ for $a, b \in k$, see e.g. [26, §1.1.6]. Henceforth, let $\boldsymbol{q}$, $\boldsymbol{t}_1, \dots, \boldsymbol{t}_m$, $\boldsymbol{s}_1, \dots, \boldsymbol{s}_m$ be algebraically independent over $\mathbf{Q}$. Let $\boldsymbol{t} = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ and $\boldsymbol{s} = (\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)$. For $a \in \mathbf{Q}[\boldsymbol{s}]$, write $\boldsymbol{q}^a := (1 + (\boldsymbol{q}-1))^a \in \mathbf{Q}[\boldsymbol{s}][\![\boldsymbol{q}-1]\!]$, and let $\boldsymbol{q}^{-\boldsymbol{s}} := (\boldsymbol{q}^{-\boldsymbol{s}_1}, \dots, \boldsymbol{q}^{-\boldsymbol{s}_m})$.

Lemma 5.1. *The homomorphism $f(\boldsymbol{q}, \boldsymbol{t}) \mapsto f(\boldsymbol{q}, \boldsymbol{q}^{-\boldsymbol{s}})$ embeds $\mathbf{Q}[\boldsymbol{q}, \boldsymbol{t}]$ into $\mathbf{Q}[\boldsymbol{s}][\![\boldsymbol{q}-1]\!]$.*

Proof. Let $0 \neq f(\boldsymbol{q}, \boldsymbol{t}) \in \mathbf{Q}[\boldsymbol{q}, \boldsymbol{t}]$. By induction, we find $r \in \mathbf{N}^m$ with $f(\boldsymbol{q}, \boldsymbol{q}^{r_1}, \dots, \boldsymbol{q}^{r_m}) \neq 0$. The binomial theorem shows that if we substitute $\boldsymbol{s}_j \leftarrow -r_j$ in the coefficients of $f(\boldsymbol{q}, \boldsymbol{q}^{-\boldsymbol{s}})$ as a series in $\boldsymbol{q}-1$, we obtain the coefficients of $f(\boldsymbol{q}, \boldsymbol{q}^{r_1}, \dots, \boldsymbol{q}^{r_m})$ as a polynomial in $\boldsymbol{q}-1$. As $f(\boldsymbol{q}, \boldsymbol{q}^{r_1}, \dots, \boldsymbol{q}^{r_m}) \neq 0$, we conclude that $f(\boldsymbol{q}, \boldsymbol{q}^{-\boldsymbol{s}}) \neq 0$. ♦

Hence, $f(\boldsymbol{q}, \boldsymbol{t}) \mapsto f(\boldsymbol{q}, \boldsymbol{q}^{-\boldsymbol{s}})$ extends to an embedding of $\mathbf{Q}(\boldsymbol{q}, \boldsymbol{t})$ into the field $\mathbf{Q}(\boldsymbol{s})(\!(\boldsymbol{q}-1)\!)$ of formal Laurent series in $\boldsymbol{q}-1$ over $\mathbf{Q}(\boldsymbol{s})$. Consider the $\mathbf{Q}$-subalgebra, $\mathcal{A}$ say, of $\mathbf{Q}(\boldsymbol{q}, \boldsymbol{t})$ generated by $\boldsymbol{q}^{\pm 1}, \boldsymbol{t}_1^{\pm 1}, \dots, \boldsymbol{t}_m^{\pm 1}$, and all $(\boldsymbol{q}^a \boldsymbol{t}^b - 1)^{-1}$ with $(0,0) \neq (a,b) \in \mathbf{Z}^{1+m}$.

Definition 5.2. Let $\mathbf{M}$ be the $\mathbf{Q}$-subalgebra of $\mathbf{Q}(\boldsymbol{q}, \boldsymbol{t})$ consisting of those $W(\boldsymbol{q}, \boldsymbol{t}) \in \mathcal{A}$ with $W(\boldsymbol{q}, \boldsymbol{q}^{-\boldsymbol{s}}) \in \mathbf{M}^\sharp := \mathbf{Q}(\boldsymbol{s})[\![\boldsymbol{q}-1]\!]$.

We henceforth regard $\mathbf{M}$ as a subalgebra of $\mathbf{M}^\sharp$ via $W(\boldsymbol{q}, \boldsymbol{t}) \mapsto W(\boldsymbol{q}, \boldsymbol{q}^{-\boldsymbol{s}})$. For added clarity, we sometimes write $\mathbf{M}(\boldsymbol{q}; \boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ instead of $\mathbf{M}$.

Notation 5.3. Let $\lfloor - \rfloor \colon \mathbf{M}^\sharp \to \mathbf{Q}(\boldsymbol{s})$ denote reduction modulo $\boldsymbol{q}-1$.

For $W^\sharp \in \mathbf{M}^\sharp$, we sometimes write $\lfloor W^\sharp(\boldsymbol{s}) \rfloor$ instead of $\lfloor W^\sharp \rfloor(\boldsymbol{s})$.

Example 5.4.

(i) $\boldsymbol{q}^{-1} = \sum_{d=0}^{\infty} (-1)^d (\boldsymbol{q}-1)^d$ and $\boldsymbol{t}_j^{-1} = \sum_{d=0}^{\infty} \binom{\boldsymbol{s}_j}{d} (\boldsymbol{q}-1)^d$ so that $\lfloor \boldsymbol{q}^{\pm 1} \rfloor = \lfloor \boldsymbol{t}_j^{\pm 1} \rfloor = 1$.

(ii) $\frac{\boldsymbol{q}-1}{\boldsymbol{q}^a \boldsymbol{t}^b - 1} \in \mathbf{M}$ and $\left\lfloor \frac{\boldsymbol{q}-1}{\boldsymbol{q}^a \boldsymbol{t}^b - 1} \right\rfloor = \frac{1}{a - \langle b, \boldsymbol{s} \rangle} = \frac{1}{a - b_1 \boldsymbol{s}_1 - \dots - b_m \boldsymbol{s}_m}$ for $(0,0) \neq (a,b) \in \mathbf{Z}^{1+m}$.

Fix a rational prime ℓ. For $v \geqslant 1$, let $U_v = \{x \in \mathbf{Z}_\ell : |x-1|_\ell \leqslant \ell^{-v}\}$, a subgroup of $\mathbf{Z}_\ell^\times$. It is well-known that each U_v admits a unique continuous $\mathbf{Z}_\ell$-module structure $U_v \times \mathbf{Z}_\ell \to U_v, (x, e) \mapsto x^e$ extending the canonical $\mathbf{Z}$-action. The following is a multivariate version of an argument from [17, (2.4)].

Proposition 5.5. *Let $W(\boldsymbol{q}, \boldsymbol{t}) \in \mathbf{M}$. There exists a finite union $\mathcal{H}$ of affine hyperplanes (defined over $\mathbf{Z}$, independently of ℓ) in $\mathbf{Q}_\ell^m$ such that $U_2 \setminus \{1\} \times \mathbf{Z}^m \setminus \mathcal{H} \to \mathbf{Q}_\ell, (x, s) \mapsto W(x, x^{-s_1}, \dots, x^{-s_m})$ extends to a continuous map $F \colon U_2 \times \mathbf{Z}_\ell^m \setminus \mathcal{H} \to \mathbf{Q}_\ell$ with $F(1, s) = \lfloor W(s) \rfloor$ for $s \in \mathbf{Z}_\ell^m \setminus \mathcal{H}$.*

Note that $\mathbf{Z}^m \setminus \mathcal{H}$ is dense in $\mathbf{Z}_\ell^m$ whence F is unique. Proposition 5.5 is a consequence of the following more technical statement.

Lemma 5.6. *Let $W(\boldsymbol{q}, \boldsymbol{t}) \in \mathbf{M}$ and $v \geqslant 1$ with $\ell^v \geqslant 3$. Suppose that*

$$\prod_{(a,b) \in \Lambda} (\boldsymbol{q}^a \boldsymbol{t}^b - 1)^{e(a,b)} \cdot W(\boldsymbol{q}, \boldsymbol{t}) \in \mathbf{Q}[\boldsymbol{q}^{\pm 1}, \boldsymbol{t}^{\pm 1}],$$

where $\Lambda = \mathbf{Z}^{1+m} \setminus \{0\}$ and $e \colon \Lambda \to \mathbf{N}_0$ has finite support.

(i) Define $\mathcal{H} = \{s \in \mathbf{Z}^m : a = \langle b, s\rangle$ *for some* $(a,b) \in \Lambda$ *with* $e(a,b) > 0\}$. *Then*

$$G_0\colon U_v\backslash\{1\}\times\mathbf{Z}^m\backslash\mathcal{H} \to \mathbf{Q}_\ell, \quad (x,s) \mapsto \prod_{(a,b)\in\Lambda} (a-\langle b,s\rangle)^{e(a,b)} \cdot W(x, x^{-s_1}, \ldots, x^{-s_m})$$

extends to a continuous map $G\colon U_v \times \mathbf{Z}_\ell^m \to \mathbf{Q}_\ell$.

(ii) $g(\boldsymbol{s}) := \prod_{(a,b)\in\Lambda}(a-\langle b,\boldsymbol{s}\rangle)^{e(a,b)}\lfloor W(\boldsymbol{s})\rfloor$ *belongs to* $\mathbf{Q}[\boldsymbol{s}]$ *and* $G(1,s)=g(s)$ *for* $s\in\mathbf{Z}_\ell^m$.

Proof. First, there exists a continuous map $h\colon U_v \times \mathbf{Z}_\ell \to \ell\mathbf{Z}_\ell$ such that (a) $x^e - 1 = e(x-1)\cdot(1+h(x,e))$ for all $x \in U_v$ and $e \in \mathbf{Z}_\ell$ and (b) $h(1,-) = 0$. Indeed, for $e \neq 0$, the ℓ-adic binomial series [9, Cor. 4.2.18] yields

$$x^e - 1 = e(x-1)\cdot\left(1 + \sum_{d=1}^{\infty}\binom{e-1}{d}\cdot\frac{(x-1)^d}{d+1}\right)$$

and for $d \geqslant 1$, we have

$$\sup_{x\in U_v, e\in\mathbf{Z}_\ell}\left|\binom{e-1}{d}\cdot\frac{(x-1)^d}{d+1}\right|_\ell \leqslant \sup_{x\in U_v}\left((d+1)\cdot|x-1|_\ell^d\right) \leqslant \frac{d+1}{3^d} < 1.$$

Let $f(\boldsymbol{q},\boldsymbol{t}) = \prod_{(a,b)\in\Lambda}(\boldsymbol{q}^a\boldsymbol{t}^b - 1)^{e(a,b)}\cdot W(\boldsymbol{q},\boldsymbol{t})$, say $f(\boldsymbol{q},\boldsymbol{t}) = \sum_{(a,b)\in\mathbf{Z}^{1+m}} c_{ab}\boldsymbol{q}^a\boldsymbol{t}^b$ with $c_{ab} \in \mathbf{Q}$, almost all of which are 0. Let $M = \sum_{(a,b)\in\Lambda} e(a,b)$. As an element of $\mathbf{M}^\sharp$, the series $W(\boldsymbol{q},\boldsymbol{q}^{-\boldsymbol{s}})$ has nonnegative $(\boldsymbol{q}-1)$-adic valuation. Hence, $u^\sharp := f(\boldsymbol{q},\boldsymbol{t})(\boldsymbol{q}-1)^{-M}$ is contained in $\mathbf{M}^\sharp$ and thus in $\mathbf{Q}[\boldsymbol{s}][\![\boldsymbol{q}-1]\!]$ since $f(\boldsymbol{q},\boldsymbol{q}^{-\boldsymbol{s}}) \in \mathbf{Q}[\boldsymbol{s}][\![\boldsymbol{q}-1]\!]$. Note that $W(\boldsymbol{q},\boldsymbol{t}) = u^\sharp\cdot\prod_{(a,b)\in\Lambda}\left(\frac{\boldsymbol{q}-1}{\boldsymbol{q}^a\boldsymbol{t}^b-1}\right)^{e(a,b)}$. Thus, $g(\boldsymbol{s})$ in part (ii) of the lemma coincides with $\lfloor u^\sharp\rfloor \in \mathbf{Q}[\boldsymbol{s}]$. The ℓ-adic binomial series shows that the map

$$U_v\backslash\{1\}\times\mathbf{Z}^m \to \mathbf{Q}_\ell, \quad (x,s)\mapsto f(x, x^{-s_1},\ldots,x^{-s_m})(x-1)^{-M}$$

admits the continuous extension

$$F\colon U_v\times\mathbf{Z}_\ell^m \to \mathbf{Q}_\ell, \quad (x,s)\mapsto \sum_{d=0}^{\infty}\left(\sum_{(a,b)\in\mathbf{Z}^{1+m}} c_{ab}\binom{a-\langle b,s\rangle}{d+M}\right)(x-1)^d$$

which satisfies $F(1,s) = \sum_{(a,b)\in\mathbf{Z}^{1+m}} c_{ab}\binom{a-\langle b,s\rangle}{M} = \lfloor u^\sharp(s)\rfloor$ for $s \in \mathbf{Z}_\ell^m$. We thus obtain an extension of G_0 of the desired form, namely

$$(x,s)\mapsto F(x,s)\cdot\prod_{(a,b)\in\Lambda}(1+h(x,a-\langle b,s\rangle))^{-e(a,b)}. \qquad \blacklozenge$$

5.2 Local zeta functions of Denef type

Throughout this subsection, let k be a number field with ring of integers $\mathfrak{o}$.

Definition 5.7. By a **system of local zeta functions** (in m variables) over k, we mean a family $\mathsf{Z} = (\mathsf{Z}_K)$ such that

(i) K ranges over all non-Archimedean local fields endowed with an embedding $k \subset K$,

(ii) each Z_K is a meromorphic function of the form

$$\mathsf{Z}_K(s_1, \dots, s_m) = W_K(q_K^{-s_1}, \dots, q_K^{-s_m})$$

for complex $s_1, \dots, s_m$ and a rational function $W_K(\boldsymbol{t}_1, \dots, \boldsymbol{t}_m) \in \mathbf{Q}(\boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$, where q_K denotes the residue field size of K, and

(iii) $\mathsf{Z}_{K_1} = \mathsf{Z}_{K_2}$ whenever non-Archimedean local fields $K_1, K_2 \supset k$ are topologically isomorphic over k.

The final condition is included to avoid set-theoretic difficulties.

Definition 5.8. Let Z and Z' be systems of local zeta functions over k. We say that Z and Z' are **equivalent** if there exists a finite set S of primes of k such that $\mathsf{Z}_K = \mathsf{Z}'_K$ whenever $\mathfrak{P}_K \cap \mathfrak{o} \notin S$, where $\mathfrak{P}_K$ is the maximal ideal of the valuation ring of K.

We now consider systems of local zeta functions given (up to equivalence) by formulae similar to those obtained by Denef for Igusa's local zeta function [13, Thm 3.1]. Thus, let I be a finite set and suppose that for each $i \in I$, we are given a variety V_i over k and a rational function $W_i(\boldsymbol{q}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m) \in \mathbf{M}$ (see Definition 5.2). In the cases of interest to us, each V_i will be quasi-projective, i.e. a locally closed (but not necessarily reduced) subscheme of projective space over k. Given these data, we obtain an associated system $\mathsf{D} = (\mathsf{D}_K)$ of local zeta functions as follows: for each prime $\mathfrak{p}$ of k and non-Archimedean local field $K \supset k$ such that the maximal ideal $\mathfrak{P}_K$ of its valuation ring $\mathfrak{O}_K$ satisfies $\mathfrak{P}_K \cap \mathfrak{o} = \mathfrak{p}$, we let

$$\mathsf{D}_K(s_1, \dots, s_m) := \sum_{i \in I} \# \bar{V}_i(\mathfrak{O}_K/\mathfrak{P}_K) \cdot W_i(q_K, q_K^{-s_1}, \dots, q_K^{-s_m}),$$

where q_K is the residue field size of K and $\bar{\cdot}$ denotes "reduction modulo $\mathfrak{p}$". A rigorous definition of "reduction modulo $\mathfrak{p}$" is a subtle matter, see [13, §2] for Denef's definition. For our purposes, the following naive approach will suffice. Namely, we choose, once and for all, for each $i \in I$, a model X_i of V_i over $\mathfrak{o}$, i.e. a separated scheme X_i of finite type over $\mathfrak{o}$ with $(X_i)_k \approx_k V_i$. Given $\mathfrak{p}$, we then define $\bar{V}_i := X_i \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\mathfrak{p})$. While this definition of $\bar{V}_i$ is not intrinsic, for a fixed rational prime ℓ, the following numbers do not depend on our choice of X_i, provided we are willing to ignore finitely many $\mathfrak{p}$:

(N1) the number of rational points of $\bar{V}_i$ over finite field extensions of $\mathfrak{o}/\mathfrak{p}$ and

(N2) the ℓ-adic Euler characteristic $\chi_c((\bar{V}_i)_{\overline{\mathfrak{o}/\mathfrak{p}}}, \mathbf{Q}_\ell)$ with compact support of $\bar{V}_i$, taken after base changing to an algebraic closure $\overline{\mathfrak{o}/\mathfrak{p}}$ of $\mathfrak{o}/\mathfrak{p}$—indeed, $\chi_c((\bar{V}_i)_{\overline{\mathfrak{o}/\mathfrak{p}}}, \mathbf{Q}_\ell) = \chi_c((V_i)_{\bar{k}}, \mathbf{Q}_\ell)$;

see e.g. [46, §4.8.2] and cf. [21, 7.6], [41, §1.3], [42, §5.3]. In particular, up to equivalence, D only depends on the collection $(V_i, W_i(\boldsymbol{q}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m))_{i \in I}$.

Remark 5.9. By Artin's comparison theorem and invariance of ℓ-adic cohomology under extensions of algebraically closed fields (see [32] for both), for almost all $\mathfrak{p}$, the Euler characteristic in (N2) coincides with the topological Euler characteristic (with or without compact support [36, A4, Prop.]) of the $\mathbf{C}$-analytic space associated with V_i for any embedding $k \subset \mathbf{C}$.

Definition 5.10. We say that a system of local zeta functions over k is of **Denef type** if it is equivalent to a system of the form D just constructed.

For the following examples of zeta functions, we do not distinguish between defining integrals (or Dirichlet series) and associated meromorphic continuations. Moreover, as we are only interested in systems of local zeta functions up to equivalence, the "explicit formulae" cited below provide the rationality properties required by Definition 5.7(ii); we note that, instead of proving rationality using model-theoretic techniques, the aforementioned explicit formulae can be modified to cover exceptional primes, see [1, §4.3].

Example 5.11. As before, let k be a number field with ring of integers $\mathfrak{o}$. We follow the conventions for local fields in Notation 2.5 with subscripts $(-)_K$ added for clarity.

(i) The prime example of a system of local zeta functions of Denef type, and the reason for our choice of terminology, is given by Igusa's local zeta function. Thus, for $f \in k[\boldsymbol{X}]$, the family $\mathsf{Z}_f = (\mathsf{Z}_{f,K})$ given by $\mathsf{Z}_{f,K}(s) := \int_{\mathfrak{O}_K^n} |f(\boldsymbol{x})|_K^s \,\mathrm{d}\mu_K(\boldsymbol{x})$ is of Denef type. For a proof, take Denef's "explicit formula" [13, Thm 3.1] and apply [13, Prop. 2.3, Thm 2.4] to pass from completions $k_\mathfrak{p}$ of k to finite extensions; see also [14, Rem. 2.3 & §3].

(ii) More generally, for non-empty finite $\boldsymbol{f} \subset k[\boldsymbol{X}]$, the family $\mathsf{Z}_{\boldsymbol{f}}$ with $\mathsf{Z}_{\boldsymbol{f},K}(s) := \int_{\mathfrak{O}_K^n} \|\boldsymbol{f}(\boldsymbol{x})\|_K^s \,\mathrm{d}\mu_K(\boldsymbol{x})$ is of Denef type. Here, one takes the formula of Veys and Zúñiga-Galindo [51, Thm 2.10] and uses stability under base change as in (i).

(iii) Let $\mathcal{A}$ be a not necessarily associative $\mathfrak{o}$-algebra which is free of finite rank d as an $\mathfrak{o}$-module. Define a family $\mathsf{Z}_\mathcal{A}$ by $\mathsf{Z}_{\mathcal{A},K}(s) := (1-q_K^{-1})^d \zeta_{\mathcal{A}_{\mathfrak{O}_K}}(s)$ (see Definition 2.1(i)). Similarly, for a free $\mathfrak{o}$-module M of rank d and a subalgebra $\mathcal{E}$ of $\mathrm{End}_\mathfrak{o}(M)$, define $\mathsf{Z}_{\mathcal{E} \curvearrowright M}$ by $\mathsf{Z}_{\mathcal{E} \curvearrowright M,K}(s) := (1-q_K^{-1})^d \zeta_{\mathcal{E}_{\mathfrak{O}_K} \curvearrowright M_{\mathfrak{O}_K}}(s)$ (see Definition 2.1(ii)). Then $\mathsf{Z}_\mathcal{A}$ and $\mathsf{Z}_{\mathcal{E} \curvearrowright M}$ are both of Denef type, see Theorem 5.16.

(iv) Taking $\mathsf{Z}_K(s_1, \dots, s_m)$ to be the integral in Proposition 3.9, we obtain a system (Z_K) of local zeta functions of Denef type. Indeed, the remarks following Theorem 3.4 show that $(1-\boldsymbol{q}^{-1})^n \mathscr{Z}^{\mathcal{C}_0, \mathcal{P}_1, \dots, \mathcal{P}_m}(\boldsymbol{q}^{-1}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m) \in \mathbf{M}$ (see Definition 5.2).

(v) Further examples of multivariate local zeta functions of Denef type are provided by [53, Thm 2.1]. For the stability under base change, one again argues as in (i), while the interpretation of rational functions as elements of $\mathbf{M}$ is similar to (iv).

(vi) Let $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$ be a half-open rational cone and let $\boldsymbol{f}_0 \subset \mathfrak{o}[\boldsymbol{X}^{\pm 1}]$ and $\boldsymbol{f}_1, \dots, \boldsymbol{f}_m \subset \mathfrak{o}[\boldsymbol{X}] = \mathfrak{o}[X_1, \dots, X_n]$ be non-empty finite sets with $0 \notin \boldsymbol{f} := \boldsymbol{f}_0 \cup \dots \cup \boldsymbol{f}_m$. Disregarding finitely many primes $\mathfrak{P}_K \cap \mathfrak{o}$ of k, if $\boldsymbol{f} := \boldsymbol{f}_0 \cup \dots \cup \boldsymbol{f}_m$ is non-degenerate relative to $\mathcal{C}_0$ (see Definition 4.2(i)), then Theorem 4.10 shows that $\mathsf{Z}^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m} := \left(\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}\right)$ is a system of local zeta functions. While $\mathsf{Z}^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}$ is indeed of Denef type, Theorem 4.10 falls short of making this explicit because the rational functions $\boldsymbol{q}^{-n}(\boldsymbol{q}-1)^{|\boldsymbol{g}|} \mathcal{Z}^{\mathcal{C}_0^\tau(\boldsymbol{g}), \mathcal{P}_1^\tau(\boldsymbol{g}), \dots, \mathcal{P}_m^\tau(\boldsymbol{g})}(\boldsymbol{q}^{-1}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ in Theorem 4.10 need not belong to $\mathbf{M}$. This "flaw" is a minor one which we will fix in Lemma 6.9.

Although we will not use it in the following, we note that $\mathsf{Z}^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}$ is a system of local zeta functions of Denef type without any non-degeneracy assumptions on $\boldsymbol{f}$, as can be shown using Remark 4.7 and arguments similar to (ii) and (iii).

5.3 Topological zeta functions as limits of local ones

Let Z be a system of local zeta functions in m variables over a number field k (see Definition 5.7). Suppose that Z is of Denef type (see Definition 5.10). For a prime $\mathfrak{p}$ of k and $f \geqslant 1$, let $k_{\mathfrak{p}}^{(f)}$ denote the unramified extension of degree f of the completion $k_{\mathfrak{p}}$ of k at $\mathfrak{p}$. Write $\mathsf{Z}_{\mathfrak{p}}(f; s_1, \dots, s_m) := \mathsf{Z}_{k_{\mathfrak{p}}^{(f)}}(s_1, \dots, s_m)$. Then, unless $\mathfrak{p}$ belongs to some finite exceptional set of primes of k, for any local field $K \supset k_{\mathfrak{p}}$ with residue class degree f over $k_{\mathfrak{p}}$, we have $\mathsf{Z}_K(s_1, \dots, s_m) = \mathsf{Z}_{\mathfrak{p}}(f; s_1, \dots, s_m)$.

Theorem 5.12 (Cf. [17, (2.4)]). *Let Z be a system of local zeta functions in m variables and of Denef type over a number field k. Then:*

(i) There exists a finite union $\mathcal{H}$ of affine hyperplanes in $\mathbf{A}^m_{\mathbf{Z}}$ such that for each rational prime ℓ and almost all primes $\mathfrak{p}$ of k (depending on ℓ and Z), there exists $d \geqslant 1$ such that

$$\mathbf{N} \times \mathbf{Z}^m \setminus \mathcal{H}(\mathbf{Z}) \to \mathbf{Q}, \quad (f, s) \mapsto \mathsf{Z}_{\mathfrak{p}}(df; s)$$

extends to a continuous map $\mathbf{Z}_\ell \times \mathbf{Z}_\ell^m \setminus \mathcal{H}(\mathbf{Z}_\ell) \to \mathbf{Q}_\ell$.

(ii) Notation as in (i). There exists a rational function $\mathsf{Z}_{\mathrm{top}}(\boldsymbol{s}) \in \mathbf{Q}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)$ such that for all ℓ and almost all primes $\mathfrak{p}$ (depending on ℓ and Z), there exists $d \geqslant 1$ such that $\lim\limits_{\substack{f \to 0,\\ f \in \mathbf{N}}} \mathsf{Z}_{\mathfrak{p}}(df; s) = \mathsf{Z}_{\mathrm{top}}(s) \in \mathbf{Q}$ for all $s \in \mathbf{Z}^m \setminus \mathcal{H}(\mathbf{Z})$, the limits being ℓ-adic.

(iii) Let D be defined in terms of k-varieties V_i and $W_i(\boldsymbol{q}, \boldsymbol{t}) \in \mathbf{M}$ as in Definition 5.10. Then $\mathsf{D}_{\mathrm{top}}(\boldsymbol{s}) = \sum_{i \in I} \chi(V_i(\mathbf{C})) \cdot \lfloor W_i(\boldsymbol{s}) \rfloor$ (see Notation 5.3).

Note that since $\mathbf{Z}^m \setminus \mathcal{H}(\mathbf{Z})$ is Zariski-dense in $\mathbf{C}^m$, the rational function $\mathsf{Z}_{\mathrm{top}}(\boldsymbol{s})$ in (ii) is uniquely determined (and independent of $\mathcal{H}$). Furthermore, by construction, $\mathsf{Z}_{\mathrm{top}}(\boldsymbol{s})$ only depends on the equivalence class of Z in the sense of Definition 5.8.

Definition 5.13. Let Z be a system of local zeta functions of Denef type over a number field. The **topological zeta function** associated with Z is the rational function $\mathsf{Z}_{\mathrm{top}}(\boldsymbol{s})$ in Theorem 5.12(ii).

This definition generalises the case of Igusa's local zeta function in [17]. Denef and Loeser apparently chose the name owing to the appearance of the topological Euler characteristics of the $\mathbf{C}$-analytic spaces $V_i(\mathbf{C})$ (defined with respect to an arbitrary embedding $k \subset \mathbf{C}$) in Theorem 5.12(iii).

The following result needed to derive Theorem 5.12 can be found in [17, (2.4)]; we include a short group-theoretic proof.

Lemma 5.14. *Let V be a separated scheme of finite type over $\mathbf{F}_q$. Let ℓ be a rational prime with $\ell \nmid q$. There exists $d \in \mathbf{N}$ such that $\mathbf{N} \to \mathbf{N}_0, f \mapsto \#V(\mathbf{F}_{q^{df}})$ extends to a continuous map $\alpha\colon \mathbf{Z}_\ell \to \mathbf{Z}_\ell$ which satisfies $\alpha(0) = \chi_c(V_{\bar{\mathbf{F}}_q}, \mathbf{Q}_\ell)$.*

Proof. We use some basic properties of ℓ-adic cohomology as e.g. explained in [32]. Let Γ be the absolute Galois group of $\mathbf{F}_q$ and let $\sigma \in \Gamma$ be the arithmetic Frobenius of $\mathbf{F}_q$. It is well-known [46, Thm 4.4] that the geometric Frobenius of V and σ^{-1} induce the same automorphism, γ_i say, of $W_i := \mathrm{H}^i_c(V_{\bar{\mathbf{F}}_q}, \mathbf{Q}_\ell)$ for every $i \geqslant 0$. Hence, by Grothendieck's trace formula, $\#V(\mathbf{F}_{q^f}) = \sum_i (-1)^i \operatorname{trace}(\gamma_i^f)$ for all $f \geqslant 1$. Since the natural action $\Gamma \to G_i := \mathrm{GL}(W_i)$ is continuous and $G_i \approx \mathrm{GL}_{\dim(W_i)}(\mathbf{Q}_\ell)$ is ℓ-adic analytic, there exists $d \in \mathbf{N}$ such that each γ_i^d is contained in a pro-ℓ subgroup of G_i. It follows that each map $\mathbf{Z} \to G_i, f \mapsto \gamma_i^{df}$ admits a continuous extension $\mathbf{Z}_\ell \to G_i$. ♦

Proof of Theorem 5.12. Let D be as in (iii) and suppose that Z is equivalent to D. Fix an arbitrary ℓ. We may choose a single $\mathcal{H}$ (independent of ℓ) such that the conclusion of Proposition 5.5 holds for all $W_i(\boldsymbol{q}, \boldsymbol{t})$ (with $\mathcal{H}(\mathbf{Z}_\ell)$ in place of $\mathcal{H}$). Let $\mathfrak{p}$ be a prime of k and let q be the size of the residue field. Suppose that $\ell \nmid q$. There exists $d \geqslant 1$ such that the conclusion of Lemma 5.14 holds for all $\bar{V}_i$ and, additionally, $|q^d - 1| \leqslant \ell^{-2}$. Part (i) follows. Next, $\lim_{f \to 0} \mathsf{D}_{\mathfrak{p}}(df; s) = \sum_{i \in I} \chi_c((\bar{V}_i)_{\overline{\mathfrak{o}/\mathfrak{p}}}, \mathbf{Q}_\ell) \cdot \lfloor W_i(s) \rfloor$ for $s \in \mathbf{Z}^m \setminus \mathcal{H}(\mathbf{Z})$. As $\chi_c((\bar{V}_i)_{\overline{\mathfrak{o}/\mathfrak{p}}}, \mathbf{Q}_\ell) = \chi(V_i(\mathbf{C}))$ for almost all $\mathfrak{p}$ (see Remark 5.9), parts (ii)–(iii) follow. ♦

Remark 5.15. As we will now explain, passing from local to topological zeta functions is compatible with suitable affine specialisations. Thus, let D be a system of local zeta functions of Denef type over a number field k, arising from varieties V_i and $W_i(\boldsymbol{q}, \boldsymbol{t}) \in \mathbf{M}(\boldsymbol{q}; \boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ as in Definition 5.10. Let $\boldsymbol{c} \in \mathbf{Z}^m$, $A \in \mathrm{M}_{m \times r}(\mathbf{Z})$ with rows $A_1, \dots, A_m$, and let $\tilde{\boldsymbol{t}} := (\tilde{\boldsymbol{t}}_1, \dots, \tilde{\boldsymbol{t}}_r)$ and $\tilde{\boldsymbol{s}} := (\tilde{\boldsymbol{s}}_1, \dots, \tilde{\boldsymbol{s}}_r)$. Suppose that

$$\prod_{(a,b) \in \Lambda} (\boldsymbol{q}^a \boldsymbol{t}^b - 1)^{e(a,b)} \cdot W_i(\boldsymbol{q}, \boldsymbol{t}) \in \mathbf{Q}[\boldsymbol{q}^{\pm 1}, \boldsymbol{t}^{\pm 1}]$$

for all $i \in I$, where $\Lambda = \mathbf{Z}^{1+m} \setminus \{0\}$ and $e\colon \Lambda \to \mathbf{N}_0$ has finite support. If $\boldsymbol{q}^{a + \langle b, c \rangle} \tilde{\boldsymbol{t}}^{bA} \neq 1$ for $e(a,b) > 0$, we may define $W'_i(\boldsymbol{q}, \tilde{\boldsymbol{t}}) := W_i(\boldsymbol{q}, \boldsymbol{q}^{c_1} \tilde{\boldsymbol{t}}^{A_1}, \dots, \boldsymbol{q}^{c_m} \tilde{\boldsymbol{t}}^{A_m})$ and it is easily verified that each $W'_i(\boldsymbol{q}, \tilde{\boldsymbol{t}})$ belongs to $\mathbf{M}(\boldsymbol{q}; \tilde{\boldsymbol{t}})$. We thus obtain a system of local zeta functions in r variables, D' say, associated with the V_i and $W'_i(\boldsymbol{q}, \tilde{\boldsymbol{t}})$. We then have

$$\mathsf{D}'_{\mathrm{top}}(\tilde{\boldsymbol{s}}_1, \dots, \tilde{\boldsymbol{s}}_r) = \mathsf{D}_{\mathrm{top}}(\langle A_1, \tilde{\boldsymbol{s}} \rangle - c_1, \dots, \langle A_m, \tilde{\boldsymbol{s}} \rangle - c_m).$$

5.4 Topological subalgebra and submodule zeta functions

Let k be a number field with ring of integers $\mathfrak{o}$. Let $\mathcal{A}$ be a not necessarily associative $\mathfrak{o}$-algebra which is free of finite rank d as an $\mathfrak{o}$-module. Let M be a free $\mathfrak{o}$-module of rank d and let $\mathcal{E}$ be an associative subalgebra of $\mathrm{End}_{\mathfrak{o}}(M)$. In Example 5.11(iii), we defined systems of local zeta functions $\mathsf{Z}_{\mathcal{A}}$ and $\mathsf{Z}_{\mathcal{E} \curvearrowright M}$ by rescaling the subalgebra and submodule zeta functions associated with $\mathcal{A}$ and $\mathcal{E}$ acting on M, respectively.

Theorem 5.16 (Cf. [20, §§2–3])**.** *$\mathsf{Z}_{\mathcal{A}}$ and $\mathsf{Z}_{\mathcal{E} \curvearrowright M}$ are both of Denef type.*

Proof. We only consider $\mathsf{Z}_{\mathcal{A}}$, the case of $\mathsf{Z}_{\mathcal{E} \curvearrowright M}$ being essentially identical. First, up to a shift $s \mapsto s + d$, Theorem 2.6 expresses $\mathsf{Z}_{\mathcal{A},K}$ in terms of the "cone integrals" studied in [20]. In particular, [20, Thm 1.4] provides explicit formulae for $\mathsf{Z}_{\mathcal{A},K}(s)$ in terms of an embedded resolution of singularities over k. Strictly speaking, [20, Thm 1.4] only covers the case $k = \mathbf{Q}, K = \mathbf{Q}_p$ but as [20, Thm 1.4] draws upon and generalises Denef's explicit formula [13, Thm 3.1] and its proof, the extension to an arbitrary number field k is straightforward. Again, as in the context of Igusa's local zeta function (see Example 5.11(i) and cf. [1, §4.2]), one deduces the desired behaviour of [20, Thm 1.4] under base change. Given the shape of their explicit formula, it only remains to show that the rational functions occurring there give rise to elements of our ring $\mathbf{M}$. This follows since, using their notation, we have $|M_k| = \dim(R_k) \leqslant |I_k|$ for $k = 0, \dots, w$. ♦

We thus obtain associated topological zeta functions (see Definition 5.13). In the univariate case, we do not distinguish between $\boldsymbol{s}$ and $\boldsymbol{s}_1$ in our notation.

Definition 5.17.

(i) The **topological subalgebra zeta function** of $\mathcal{A}$ is

$$\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s}) := \mathsf{Z}_{\mathcal{A},\mathrm{top}}(\boldsymbol{s}).$$

(ii) The **topological submodule zeta function** of $\mathcal{E}$ acting on M is

$$\zeta_{\mathcal{E} \curvearrowright M,\mathrm{top}}(\boldsymbol{s}) := \mathsf{Z}_{\mathcal{E} \curvearrowright M,\mathrm{top}}(\boldsymbol{s}).$$

We furthermore define the **topological ideal zeta function** $\zeta^{\triangleleft}_{\mathcal{A},\mathrm{top}}(\boldsymbol{s})$ of $\mathcal{A}$ in the evident way using Remark 2.2(ii).

Remark 5.18. The topological subalgebra zeta function of $\mathcal{A}$ as defined by du Sautoy and Loeser [21, §8] coincides with the function $d! \cdot \zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s} + d)$ in our notation. The author feels that avoiding this transformation yields a more natural definition. We note that the factor $d!$ seems to have been omitted from the examples given in [21, §9].

Recall that we use subscripts to denote base change.

Proposition 5.19. *Let $\bar{k}$ be an algebraic closure of k. Let $k' \subset \bar{k}$ be another number field and let $\mathfrak{o}'$ be its ring of integers. Let $\mathcal{A}'$ be a not necessarily associative $\mathfrak{o}'$-algebra which is free of finite rank as an $\mathfrak{o}'$-module.*

(i) If $k = k'$ and $\mathcal{A}_k \approx_k \mathcal{A}'_k$, then $\mathsf{Z}_{\mathcal{A}}$ and $\mathsf{Z}_{\mathcal{A}'}$ are equivalent (see Definition 5.8).

(ii) If $\mathcal{A}_{\bar{k}} \approx_{\bar{k}} \mathcal{A}'_{\bar{k}}$, then $\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s}) = \zeta_{\mathcal{A}',\mathrm{top}}(\boldsymbol{s})$.

Corresponding statements hold for $\mathsf{Z}_{\mathcal{E}\curvearrowright M}$ and $\zeta_{\mathcal{E}\curvearrowright M,\mathrm{top}}(\boldsymbol{s})$.

Proof.

(i) Every k-isomorphism $\mathcal{A}_k \to \mathcal{A}'_k$ is defined over $\mathfrak{o}_{\mathfrak{p}}$ for almost all primes $\mathfrak{p}$ of k.

(ii) Every $\bar{k}$-isomorphism $\mathcal{A}_{\bar{k}} \to \mathcal{A}'_{\bar{k}}$ is defined over some common finite extension of k and k' and $\mathsf{Z}_{\mathcal{A},\mathrm{top}}(\boldsymbol{s})$ is easily seen to be invariant under finite extensions of k. ♦

Note that by the weak Nullstellensatz, we may replace $\bar{k}$ by any algebraically closed extension of k in Proposition 5.19(ii).

Remark 5.20. Suppose that there exists $W(\boldsymbol{q},\boldsymbol{t}) \in \mathbf{M}$ such that $\mathsf{Z}_{\mathcal{A}}$ is equivalent to $\big(W(q_K, q_K^{-s})\big)_K$. In view of [21, 7.11–7.15], it is presently unclear if the existence of such a $W(\boldsymbol{q},\boldsymbol{t})$ is equivalent to uniformity of the subalgebra zeta functions of $\mathcal{A}$ in the sense of [24, §1.2.4] (for $k = \mathbf{Q}$). Given the existence of $W(\boldsymbol{q},\boldsymbol{t})$, the informal approach for reading off $\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s})$ from $\zeta_{\mathcal{A}\otimes_{\mathfrak{o}}\mathfrak{o}_{\mathfrak{p}}}(s)$ given in the introduction is actually correct.

Nilpotent groups. Let G be a torsion-free finitely generated nilpotent group. Choose an arbitrary full Lie lattice $\mathcal{L}$ inside $\mathfrak{L}(G)$. By Theorem 2.4, it is sensible to define the **topological subgroup zeta function** of G as $\zeta_{G,\mathrm{top}}(\boldsymbol{s}) := \zeta_{\mathcal{L},\mathrm{top}}(\boldsymbol{s})$ and the **topological normal subgroup zeta function** of G as $\zeta^{\triangleleft}_{G,\mathrm{top}}(\boldsymbol{s}) := \zeta^{\triangleleft}_{\mathcal{L},\mathrm{top}}(\boldsymbol{s})$. By Proposition 5.19, these rational functions are actually invariants of the $\mathbf{C}$-algebra $\mathfrak{L}(G) \otimes_{\mathbf{Q}} \mathbf{C}$.

6 Non-degeneracy II: computing topological zeta functions

As our second main result (Theorem 6.7), assuming that the set $\boldsymbol{f}$ of Laurent polynomials in Theorem 4.10 is *globally* non-degenerate (see Definition 4.2(ii)), we give an explicit convex-geometric formula for the topological zeta function associated with the $\mathfrak{P}$-adic integral $\mathsf{Z}_K^{\mathcal{C}_0,\boldsymbol{f}_0,\dots,\boldsymbol{f}_m}(s_1,\dots,s_m)$ in Definition 4.6.

6.1 Splitting off torus factors and explicit formulae for Euler characteristics

Let $(f_i)_{i\in I}$ be a finite collection of non-zero Laurent polynomials from $\mathbf{C}[X_1^{\pm1},\dots,X_n^{\pm1}]$. For $J \subset I$, let $V_J = \{\boldsymbol{u} \in \mathbf{T}^n(\mathbf{C}) : \forall i \in I\colon f_i(\boldsymbol{u}) = 0 \iff i \in J\}$. Such Boolean combinations of subvarieties of complex tori are $\mathbf{C}$-analogues of the $\bar{V}_{\boldsymbol{g}}^{\tau}$ in Definition 4.9(iii) (for fixed τ). Write $d = \dim(\mathrm{New}(\prod_{i\in I} f_i))$. Lemma 6.1(i) shows that there are $\tilde{f}_i \in \mathbf{C}[X_1^{\pm1},\dots,X_d^{\pm1}]$ such that $V_J \approx \tilde{V}_J \times \mathbf{T}^{n-d}(\mathbf{C})$ for all $J \subset I$, where $\tilde{V}_J = \{\boldsymbol{u} \in \mathbf{T}^d(\mathbf{C}) : \forall i \in I\colon \tilde{f}_i(\boldsymbol{u}) = 0 \iff i \in J\}$. For globally non-degenerate $(f_i)_{i\in I}$, using the Bernstein-Khovanskii-Kushnirenko Theorem (Theorem 6.3), in Proposition 6.5, we then derive explicit formulae for the Euler characteristics $\chi(\tilde{V}_J)$ in terms of the Newton polytopes $(\mathrm{New}(f_i))_{i\in I}$. Part (i) of the following lemma is implicit in the proof of [43, Prop. 2.5] while (ii) is similar to [43, Prop. 5.1].

Lemma 6.1. *Let k be any field and let $\boldsymbol{f} = (f_i)_{i\in I}$ be a finite family of non-zero elements of $k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Let $\mathcal{N} = \operatorname{New}\bigl(\prod_{i\in I} f_i\bigr)$ and let $\mathcal{C}_0 \subset \mathbf{R}^n$ be a half-open rational cone. Let $\operatorname{GL}_n(\mathbf{Z})$ act on $k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ via $(f, A) \mapsto f^A$ as in §3.2 (with $X_i = \lambda_i = \xi_i$).*

(i) Write $d := \dim(\mathcal{N})$. For $i \in I$, choose $\alpha_i \in \operatorname{supp}(f_i)$. Then there exists $A \in \operatorname{GL}_n(\mathbf{Z})$ such that $(\boldsymbol{X}^{-\alpha_i} f_i)^A \in k[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ for all $i \in I$.

(ii) Let $A \in \operatorname{GL}_n(\mathbf{Z})$. Write $\boldsymbol{f}^A := (f_i^A)_{i\in I}$. Then $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$ if and only if $\boldsymbol{f}^A$ is non-degenerate relative to $\mathcal{C}_0(A^{-1})^\top$.

(iii) Let $\tau \subset \mathcal{N}$ be a $\mathcal{C}_0$-visible face (see Definition 3.1) and define f_i^τ as in §4.2. Then $\operatorname{New}(\prod_{i\in I} f_i^\tau) = \tau$. If $\boldsymbol{f}$ is non-degenerate relative to $\mathcal{C}_0$, then so is $(f_i^\tau)_{i\in I}$.

Proof.

(i) Let $\hat{f}_i = \boldsymbol{X}^{-\alpha_i} f_i$. If $(\mathcal{P}_i)_{i\in I}$ is a finite family of polytopes in $\mathbf{R}^n$ with $0 \in \bigcap_{i\in I} \mathcal{P}_i$, then $\bigcup_{i\in I} \mathcal{P}_i$ and $\sum_{i\in I} \mathcal{P}_i$ span the same subspace of $\mathbf{R}^n$. Let $\mathcal{P}_i := \operatorname{New}(\hat{f}_i)$ so that $\sum_{i\in I} \mathcal{P}_i = \operatorname{New}\bigl(\prod_{i\in I} \hat{f}_i\bigr)$ has dimension d. We conclude that the $\mathbf{Z}$-submodule, M say, of $\mathbf{Z}^n$ generated by $\bigcup_{i\in I} \operatorname{supp}(\hat{f}_i)$ has rank d. The claim follows since M is contained in a direct summand of $\mathbf{Z}^n$ of the same rank, namely $\langle M\rangle_{\mathbf{Q}} \cap \mathbf{Z}^n$.

(ii) It suffices to prove one implication. Let $\boldsymbol{h} = (h_1, \dots, h_r)$ for $h_1, \dots, h_r \in k[\boldsymbol{X}^{\pm 1}] = k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Using §3.2 (and a base change to k), each $A \in \operatorname{GL}_n(\mathbf{Z})$ induces a k-automorphism $A\colon \mathbf{T}_k^n(B) \to \mathbf{T}_k^n(B)$ for every commutative k-algebra B. Taking $B = k[\boldsymbol{X}^{\pm 1}]$, a simple calculation reveals $\operatorname{diag}(A\boldsymbol{X}) \cdot A \cdot \operatorname{diag}(\boldsymbol{X}^{-1})$ to be the Jacobian matrix of $A\boldsymbol{X} = (\boldsymbol{X}^{A_1}, \dots, \boldsymbol{X}^{A_n})$, where $A_1, \dots, A_n$ denote the rows of A. As $\boldsymbol{h}^A = (h_1^A, \dots, h_r^A) = (h_1(A\boldsymbol{X}), \dots, h_r(A\boldsymbol{X}))$, we see that the Jacobian matrix of $\boldsymbol{h}^A$ is given by

$$(\boldsymbol{h}^A)' = \boldsymbol{h}'(A\boldsymbol{X}) \operatorname{diag}(A\boldsymbol{X}) A \operatorname{diag}(\boldsymbol{X}^{-1})$$

whence $(\boldsymbol{h}^A)'(\boldsymbol{u})$ and $h'(A\boldsymbol{u})$ have the same rank for all $\boldsymbol{u} \in \mathbf{T}^n(\bar{k})$, where $\bar{k}$ is an algebraic closure of k. Finally, $\operatorname{in}_\omega(f^A) = (\operatorname{in}_{\omega A^\top}(f))^A$ for all $\omega \in \mathbf{R}^n$ and $f \in k[\boldsymbol{X}^{\pm 1}]$. By applying what has been said about general $\boldsymbol{h}$ to the various families $(\operatorname{in}_{\omega A^\top}(f_j))_{j\in J}$ for $\omega \in \mathcal{C}_0(A^{-1})^\top$ and $J \subset I$, the "only if" part follows.

(iii) Let $\omega \in \mathcal{C}_0 \cap \operatorname{N}_\tau(\mathcal{N})$ be arbitrary. The first part follows from

$$\begin{aligned}\operatorname{New}\Bigl(\prod_{i\in I} f_i^\tau\Bigr) &= \operatorname{New}\Bigl(\prod_{i\in I} \operatorname{in}_\omega(f_i)\Bigr) = \sum_{i\in I} \operatorname{New}(\operatorname{in}_\omega(f_i)) = \sum_{i\in I} \operatorname{face}_\omega(\operatorname{New}(f_i)) = \\ &= \operatorname{face}_\omega\Bigl(\sum_{i\in I} \operatorname{New}(f_i)\Bigr) = \operatorname{face}_\omega(\mathcal{N}) = \tau.\end{aligned}$$

For the final statement, by combining [48, Eqns (2.3),(2.5)], for $\omega' \in \mathbf{R}^n$, we obtain $\operatorname{in}_{\omega'}(f_i^\tau) = \operatorname{in}_{\omega+\varepsilon\omega'}(f_i)$ for sufficiently small $\varepsilon > 0$. Hence, given $\omega' \in \mathcal{C}_0$, there exists $\omega'' \in \mathcal{C}_0$ with $\operatorname{in}_{\omega'}(f_i^\tau) = \operatorname{in}_{\omega''}(f_i) = f_i^{\tau'}$ for all $i \in I$, where $\tau' := \operatorname{face}_{\omega''}(\mathcal{N})$. ♦

Let Vol^n denote the Lebesgue measure on $\mathbf{R}^n$. Recall that the **mixed volume** of a collection $(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ of non-empty polytopes in $\mathbf{R}^n$ is given by

$$\mathrm{MV}^n(\mathcal{Q}_1, \ldots, \mathcal{Q}_n) = \frac{(-1)^n}{n!} \sum_{\varnothing \neq I \subset \{1,\ldots,n\}} (-1)^{|I|} \cdot \mathrm{Vol}^n\left(\sum_{i \in I} \mathcal{Q}_i\right);$$

see [10, Ch. 7, §4] for details but note that we use the *normalised* mixed volume which satisfies $\mathrm{MV}^n(\mathcal{Q}, \ldots, \mathcal{Q}) = \mathrm{Vol}^n(\mathcal{Q})$ for every non-empty polytope $\mathcal{Q} \subset \mathbf{R}^n$. The function MV^n is symmetric and invariant under translation in each argument.

Definition 6.2. Given non-empty polytopes $\mathcal{P}_1, \ldots, \mathcal{P}_r \subset \mathbf{R}^n$, let

$$\kappa^n(\mathcal{P}_1, \ldots, \mathcal{P}_r) := (-1)^{n+r} n! \sum_{\substack{c_1 + \cdots + c_r = n, \\ c_1, \ldots, c_r \in \mathbf{N}}} \mathrm{MV}^n(\underbrace{\mathcal{P}_1, \ldots, \mathcal{P}_1}_{c_1}, \ldots, \underbrace{\mathcal{P}_r, \ldots, \mathcal{P}_r}_{c_r}).$$

Note that in the special case $r = 0$, we have $\kappa^n() = 0$ for $n > 0$ and $\kappa^0() = 1$.

Theorem 6.3 ([34, §3, Thm 2])**.** *Let $f_1, \ldots, f_r \in \mathbf{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ be non-zero and suppose that $(f_1, \ldots, f_r)$ is $\cap$-non-degenerate (see Definition 4.5). Define $V = \{\boldsymbol{u} \in \mathbf{T}^n(\mathbf{C}) : f_1(\boldsymbol{u}) = \cdots = f_r(\boldsymbol{u}) = 0\}$. Then the topological Euler characteristic of V is given by $\chi(V) = \kappa^n(\mathrm{New}(f_1), \ldots, \mathrm{New}(f_r))$.*

As MV^n is symmetric, we can naturally extend the domain of κ^n to include arbitrary (unordered) finite families of polytopes in $\mathbf{R}^n$. If $\dim(\mathcal{P}_1 + \cdots + \mathcal{P}_r) < n$ in Definition 6.2, then we necessarily have $\kappa^n(\mathcal{P}_1, \ldots, \mathcal{P}_r) = 0$. We now consider a relative version of κ^n that allows us to extract useful information in such low-dimensional situations. Thus, for a d-dimensional linear subspace $U \leqslant \mathbf{R}^n$ defined over $\mathbf{Z}$, let Vol^U be the measure on U obtained from Vol^d by identifying U and $\mathbf{R}^d$ via an arbitrary choice of a $\mathbf{Z}$-isomorphism $U \cap \mathbf{Z}^n \approx \mathbf{Z}^d$. For non-empty polytopes $\mathcal{Q}_1, \ldots, \mathcal{Q}_d, \mathcal{P}_1, \ldots, \mathcal{P}_r$ contained in U, define $\mathrm{MV}^U(\mathcal{Q}_1, \ldots, \mathcal{Q}_d)$ and $\kappa^U(\mathcal{P}_1, \ldots, \mathcal{P}_r)$ in the evident way. Let $\boldsymbol{\mathcal{P}} = (\mathcal{P}_i)_{i \in I}$ be a finite family of non-empty lattice polytopes in $\mathbf{R}^n$. For each $i \in I$, choose an arbitrary point $\boldsymbol{x}_i \in \mathcal{P}_i \cap \mathbf{Z}^n$. Define $\mathcal{L}(\boldsymbol{\mathcal{P}})$ to be the linear subspace of $\mathbf{R}^n$ associated with the affine hull of $\sum_{i \in I} \mathcal{P}_i$ within $\mathbf{R}^n$. Hence, $\mathcal{L}(\boldsymbol{\mathcal{P}}) = \left\langle \sum_{i \in I} (\mathcal{P}_i - \boldsymbol{x}_i) \right\rangle_{\mathbf{R}} = \left\langle \bigcup_{i \in I} (\mathcal{P}_i - \boldsymbol{x}_i) \right\rangle_{\mathbf{R}}$ and $\dim\bigl(\mathcal{L}(\boldsymbol{\mathcal{P}})\bigr) = \dim\bigl(\sum_{i \in I} \mathcal{P}_i\bigr)$.

Definition 6.4. Given a finite family $\boldsymbol{\mathcal{P}} = (\mathcal{P}_i)_{i \in I}$ of non-empty lattice polytopes in $\mathbf{R}^n$ and a subset $J \subset I$, let $\kappa_J^{\mathrm{rel}}(\boldsymbol{\mathcal{P}}) := \sum\limits_{J \subset T \subset I} (-1)^{|T|+|J|} \cdot \kappa^{\mathcal{L}(\boldsymbol{\mathcal{P}})}\bigl(\mathcal{P}_t - \boldsymbol{x}_t\bigr)_{t \in T}$.

By translation invariance of mixed volumes, $\kappa_J^{\mathrm{rel}}(\boldsymbol{\mathcal{P}})$ does not depend on our choices of the $\boldsymbol{x}_i$. For a family $\boldsymbol{f} = (f_i)_{i \in I}$ of Laurent polynomials, write $\mathrm{New}(\boldsymbol{f}) := \bigl(\mathrm{New}(f_i)\bigr)_{i \in I}$.

Proposition 6.5. *Let $\boldsymbol{f} = (f_i)_{i \in I}$ be a finite family of non-zero $f_i \in \mathbf{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Write $\mathcal{N} = \mathrm{New}(\prod_{i \in I} f_i)$ and $d = \dim(\mathcal{N})$. Let $\alpha_i \in \mathrm{supp}(f_i)$ and $A \in \mathrm{GL}_n(\mathbf{Z})$ be as in Lemma 6.1(i). Write $\tilde{f}_i := (\boldsymbol{X}^{-\alpha_i} f_i)^A \in \mathbf{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ for $i \in I$. For $J \subset I$, let $\tilde{V}_J = \{\boldsymbol{u} \in \mathbf{T}^d(\mathbf{C}) : \forall i \in I \colon \tilde{f}_i(\boldsymbol{u}) = 0 \iff i \in J\}$. Suppose that $\boldsymbol{f}$ is globally non-degenerate in the sense of Definition 4.2(ii). Then $\chi(\tilde{V}_J) = \kappa_J^{\mathrm{rel}}(\mathrm{New}(\boldsymbol{f}))$.*

Proof. For $T \subset I$, let $\tilde{W}_T = \{\boldsymbol{u} \in \mathbf{T}^d(\mathbf{C}) : \forall t \in T\colon \tilde{f}_t(\boldsymbol{u}) = 0\}$. The additivity of topological Euler characteristics for (the associated analytic spaces of) $\mathbf{C}$-varieties (via $\chi(\,\cdot\,) = \chi_c(\,\cdot\,)$; cf. Remark 5.9) and the inclusion-exclusion principle yield $\chi(\tilde{V}_J) = \sum_{J\subset T\subset I}(-1)^{|T|+|J|} \cdot \chi(\tilde{W}_T)$. By Remark 4.3(ii) and Lemma 6.1(ii), $(\tilde{f}_i)_{i\in I}$ is globally non-degenerate. Theorem 6.3 therefore implies that $\chi(\tilde{W}_T) = \kappa^d(\mathrm{New}(\tilde{f}_t)_{t\in T})$ for $T \subset I$. Next, we have $\mathcal{L}(\mathrm{New}(\boldsymbol{f}))\cdot A = \mathbf{R}^d \times \{0\}^{n-d} \subset \mathbf{R}^n$ and hence

$$\kappa^d(\mathrm{New}(\tilde{f}_t)_{t\in T}) = \kappa^{\mathcal{L}(\mathrm{New}(\boldsymbol{f}))}(\mathrm{New}(f_t) - \alpha_t)_{t\in T}$$

whence

$$\chi(\tilde{V}_J) = \sum_{J\subset T\subset I} (-1)^{|T|+|J|} \cdot \kappa^{\mathcal{L}(\mathrm{New}(\boldsymbol{f}))}(\mathrm{New}(f_t) - \alpha_t)_{t\in T} = \kappa_J^{\mathrm{rel}}(\mathrm{New}(\boldsymbol{f})). \qquad \blacklozenge$$

6.2 Topological evaluation of non-degenerate $\mathfrak{P}$-adic integrals

Using §6.1, we now rewrite the formulae provided by Theorem 4.10 using the language of systems of local zeta functions of Denef type developed in §5.2. As the main result of this section, we then obtain Theorem 6.7, the topological counterpart of Theorem 4.10.

As in §4.3, let k be a number field with ring of integers $\mathfrak{o}$, let $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$ be a half-open rational cone, and let $\boldsymbol{f}_0 \subset \mathfrak{o}[\boldsymbol{X}^{\pm 1}]$ and $\boldsymbol{f}_1, \dots, \boldsymbol{f}_m \subset \mathfrak{o}[\boldsymbol{X}] = \mathfrak{o}[X_1, \dots, X_n]$ be non-empty finite sets with $0 \notin \boldsymbol{f} := \boldsymbol{f}_0 \cup \dots \cup \boldsymbol{f}_m$.

Definition 6.6. For a $\mathcal{C}_0$-visible face τ of $\mathcal{N} := \mathrm{New}(\prod \boldsymbol{f})$ and a subset $\boldsymbol{g} \subset \boldsymbol{f}$, let $\mathscr{Z}^{\mathcal{C}_0^\tau(\boldsymbol{g}),\mathcal{P}_1^\tau(\boldsymbol{g}),\dots,\mathcal{P}_m^\tau(\boldsymbol{g})}(\xi_0, \dots, \xi_m) \in \mathbf{Q}(\xi_0, \dots, \xi_m)$ be as in Theorem 4.10 and define

$$W_{\boldsymbol{g}}^\tau(\boldsymbol{q}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m) := \boldsymbol{q}^{-n}(\boldsymbol{q} - 1)^{n-\dim(\tau)+|\boldsymbol{g}|} \cdot \mathscr{Z}^{\mathcal{C}_0^\tau(\boldsymbol{g}),\mathcal{P}_1^\tau(\boldsymbol{g}),\dots,\mathcal{P}_m^\tau(\boldsymbol{g})}(\boldsymbol{q}^{-1}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m).$$

We will see in Lemma 6.9(i) that each $W_{\boldsymbol{g}}^\tau(\boldsymbol{q}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ belongs to the ring $\mathbf{M}$ from Definition 5.2.

Theorem 6.7. *Let k be a number field with ring of integers $\mathfrak{o}$. Let $\mathcal{C}_0 \subset \mathbf{R}^n_{\geqslant 0}$ be a half-open rational cone and let $\boldsymbol{f}_0 \subset \mathfrak{o}[\boldsymbol{X}^{\pm 1}]$ and $\boldsymbol{f}_1, \dots, \boldsymbol{f}_m \subset \mathfrak{o}[\boldsymbol{X}] = \mathfrak{o}[X_1, \dots, X_n]$ be non-empty finite sets with $0 \notin \boldsymbol{f} := \boldsymbol{f}_0 \cup \dots \cup \boldsymbol{f}_m$. Define $\mathsf{Z}_K^{\mathcal{C}_0,\boldsymbol{f}_0,\dots,\boldsymbol{f}_m}(s_1, \dots, s_m)$ as in Definition 4.6. For a $\mathcal{C}_0$-visible face τ of $\mathcal{N} := \mathrm{New}(\prod \boldsymbol{f})$ (see Definition 3.1 and §4.1) and a subset $\boldsymbol{g} \subset \boldsymbol{f}$, define a rational function $W_{\boldsymbol{g}}^\tau(\boldsymbol{q}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ as in Definition 6.6. As in §5.1, let $\lfloor W(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)\rfloor$ denote the formal reduction of $W(\boldsymbol{q}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ modulo $\boldsymbol{q} - 1$ obtained by expanding $W(\boldsymbol{q}, \boldsymbol{q}^{-\boldsymbol{s}_1}, \dots, \boldsymbol{q}^{-\boldsymbol{s}_m})$ using the formal binomial series. For $f \in \boldsymbol{f}$ and a $\mathcal{C}_0$-visible face $\tau \subset \mathcal{N}$, let f^τ be as in §4.3. Define κ^{rel} as in Definition 6.4.*

Suppose that $\boldsymbol{f}$ is globally non-degenerate in the sense of Definition 4.2(ii). Then the system of local zeta functions $\mathsf{Z}^{\mathcal{C}_0,\boldsymbol{f}_0,\dots,\boldsymbol{f}_m} := \left(\mathsf{Z}_K^{\mathcal{C}_0,\boldsymbol{f}_0,\dots,\boldsymbol{f}_m}(s_1, \dots, s_m)\right)$ is of Denef type (see Definition 5.10 and Example 5.11(vi)) and the associated topological zeta function in the sense of Definition 5.13 is given by

$$\mathsf{Z}_{\mathrm{top}}^{\mathcal{C}_0,\boldsymbol{f}_0,\dots,\boldsymbol{f}_m}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m) = \sum_{\substack{\boldsymbol{g}\subset\boldsymbol{f},\\ \tau\subset\mathcal{N}}} \kappa_{\boldsymbol{g}}^{\mathrm{rel}}\big(\mathrm{New}(f^\tau)\big)_{f\in\boldsymbol{f}} \cdot \lfloor W_{\boldsymbol{g}}^\tau(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)\rfloor \in \mathbf{Q}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m),$$

the sum being taken over all $\mathcal{C}_0$-visible faces τ of $\mathcal{N}$.

Remark 6.8. In the spirit of §4.4, we may regard Theorem 6.7 as a generalisation of the combinatorial formula for the (global) topological zeta function of a globally non-degenerate polynomial given by Denef and Loeser [17, Thm 5.3(i)]. In fact, by applying the approach explained in the following to the explicit formula of Denef and Hoornaert [16, Thm 4.2], we may recover the formula of Denef and Loeser verbatim.

We will now embark upon a proof of Theorem 6.7. First, define half-open cones $\mathcal{C}_0^\tau(\boldsymbol{g})$ and polytopes $\mathcal{P}_j^\tau(\boldsymbol{g})$ as in Definition 4.9(i)–(ii). For a $\mathcal{C}_0$-visible face $\tau \subset \mathcal{N} := \mathrm{New}(\prod \boldsymbol{f})$ and a subset $\boldsymbol{g} \subset \boldsymbol{f}$, let

$$V_{\boldsymbol{g}}^\tau := \mathrm{Spec}\left(\frac{k[\boldsymbol{X}^{\pm 1}]}{\langle g^\tau : g \in \boldsymbol{g}\rangle}\right) \setminus \mathrm{Spec}\left(\frac{k[\boldsymbol{X}^{\pm 1}]}{\langle \prod_{f\in \boldsymbol{f}\setminus \boldsymbol{g}} f^\tau\rangle}\right) \subset \mathbf{T}_k^n;$$

note that $V_{\boldsymbol{g}}^\tau \approx_k \mathrm{Spec}\Big(\frac{k[X_1^{\pm 1},\dots,X_n^{\pm 1},Y]}{\langle g^\tau : g\in \boldsymbol{g}\rangle + \langle 1 - Y\prod_{f\in\boldsymbol{f}\setminus\boldsymbol{g}} f^\tau\rangle}\Big)$ is affine. We see that, denoting reduction modulo $\mathfrak{p}$ by $\bar{\cdot}$ as in §5.2, for almost all primes $\mathfrak{p}$ of k and all field extensions $\mathfrak{k} \supset \mathfrak{o}/\mathfrak{p}$, the set $\bar{V}_{\boldsymbol{g}}^\tau(\mathfrak{k})$ coincides with its namesake in Definition 4.9(iii).

For each $\mathcal{C}_0$-visible face $\tau \subset \mathcal{N}$, using Lemma 6.1(i),(iii), choose $A^\tau \in \mathrm{GL}_n(\mathbf{Z})$ and a point $\alpha^\tau(f) \in \mathrm{supp}(f^\tau)$ for each $f \in \boldsymbol{f}$ such that $(\boldsymbol{X}^{-\alpha^\tau(f)} f^\tau)^{A^\tau} \in k\Big[X_1^{\pm 1}, \dots, X_{\dim(\tau)}^{\pm 1}\Big]$ for all $f \in \boldsymbol{f}$. Having made these choices, for $\boldsymbol{g} \subset \boldsymbol{f}$, we define

$$\widetilde{V}_{\boldsymbol{g}}^\tau := \mathrm{Spec}\left(\frac{k\Big[X_1^{\pm 1},\dots,X_{\dim(\tau)}^{\pm 1}\Big]}{\Big\langle (\boldsymbol{X}^{-\alpha^\tau(g)} g^\tau)^{A^\tau} : g \in \boldsymbol{g}\Big\rangle}\right) \setminus \mathrm{Spec}\left(\frac{k\Big[X_1^{\pm 1},\dots,X_{\dim(\tau)}^{\pm 1}\Big]}{\Big\langle \prod_{f\in\boldsymbol{f}\setminus\boldsymbol{g}} (\boldsymbol{X}^{-\alpha^\tau(f)} f^\tau)^{A^\tau}\Big\rangle}\right) \subset \mathbf{T}_k^{\dim(\tau)}.$$

Lemma 6.9.

(i) $W_{\boldsymbol{g}}^\tau(\boldsymbol{q}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m) \in \mathbf{M}$ *for all* $\mathcal{C}_0$*-visible faces* $\tau \subset \mathcal{N}$ *and subsets* $\boldsymbol{g} \subset \boldsymbol{f}$.

(ii) Notation as in Theorem 4.10. Suppose that $\boldsymbol{f}$ *is non-degenerate relative to* $\mathcal{C}_0$*. If* $K \supset k$ *is a non-Archimedean local field, then, unless* $\mathfrak{P} \cap \mathfrak{o}$ *belongs to some finite exceptional set, we have*

$$\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m) = \sum_{\substack{\boldsymbol{g}\subset\boldsymbol{f},\\ \tau\subset\mathcal{N}}} \#\bar{\tilde{V}}_{\boldsymbol{g}}^\tau(\mathfrak{O}/\mathfrak{P}) \cdot W_{\boldsymbol{g}}^\tau(q, q^{-s_1}, \dots, q^{-s_m})$$

for $s_1, \dots, s_m \in \mathbf{C}$ *with* $\mathrm{Re}(s_j) \geqslant 0$.

In particular, the system of local zeta functions $\mathsf{Z}^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}$ *consisting of the meromorphic continuations of the* $\mathsf{Z}_K^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(s_1, \dots, s_m)$ *is of Denef type and*

$$\mathsf{Z}_{\mathrm{top}}^{\mathcal{C}_0, \boldsymbol{f}_0, \dots, \boldsymbol{f}_m}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m) = \sum_{\substack{\boldsymbol{g}\subset\boldsymbol{f},\\ \tau\subset\mathcal{N}}} \chi(\tilde{V}_{\boldsymbol{g}}^\tau(\mathbf{C})) \cdot \lfloor W_{\boldsymbol{g}}^\tau(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m) \rfloor.$$

Proof.

(i) For a $\mathcal{C}_0$-visible face $\tau \subset \mathcal{N}$ and $\boldsymbol{g} \subset \boldsymbol{f}$, we have $\mathcal{C}_0^\tau(\boldsymbol{g}) \subset \mathrm{N}_\tau(\mathcal{N}) \times \mathbf{R}_{>0}^{\boldsymbol{g}}$ and hence $\dim(\mathcal{C}_0^\tau(\boldsymbol{g})) \leqslant n - \dim(\tau) + |\boldsymbol{g}|$. The claim follows from the definition of $\mathscr{Z}^{\mathcal{C}_0^\tau(\boldsymbol{g}),\mathcal{P}_1^\tau(\boldsymbol{g}),\dots,\mathcal{P}_m^\tau(\boldsymbol{g})}(\boldsymbol{q}^{-1}, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ (see Definition 3.6), the remarks following Theorem 3.4, and §3.2.

(ii) The first part follows from Theorem 4.10 and $V_{\boldsymbol{g}}^\tau \approx_k \widetilde{V}_{\boldsymbol{g}}^\tau \times_k \mathbf{T}_k^{n-\dim(\tau)}$. For the remainder, we additionally invoke (i) and Theorem 5.12(iii). Strictly speaking, Theorem 4.10 does not prove rationality of $\mathsf{Z}_K^{\mathcal{C}_0,\boldsymbol{f}_0,\dots,\boldsymbol{f}_m}(s_1,\dots,s_m)$ when $\mathfrak{P} \cap \mathfrak{o}$ is an exceptional prime, but see the comments after Definition 5.10 and note that exceptional primes are irrelevant for our purposes anyway. ♦

Proof of Theorem 6.7. Fix a $\mathcal{C}_0$-visible face $\tau \subset \mathcal{N}$ and $\boldsymbol{g} \subset \boldsymbol{f}$. Let $d = \dim(\tau)$. For $f \in \boldsymbol{f}$, write $\tilde{f}^\tau := (\boldsymbol{X}^{-\alpha^\tau(f)} f^\tau)^{A^\tau}$. By Lemma 6.1(iii), $(f^\tau)_{f\in\boldsymbol{f}}$ is globally non-degenerate. Since $\widetilde{V}_{\boldsymbol{g}}^\tau(\mathbf{C}) = \{\boldsymbol{u} \in \mathbf{T}^d(\mathbf{C}) : \forall f \in \boldsymbol{f} \colon \tilde{f}^\tau(\boldsymbol{u}) = 0 \iff f \in \boldsymbol{g}\}$, Proposition 6.5 shows that $\chi(\widetilde{V}_{\boldsymbol{g}}^\tau(\mathbf{C})) = \kappa_{\boldsymbol{g}}^{\mathrm{rel}}\big(\mathrm{New}(f^\tau)\big)_{f\in\boldsymbol{f}}$. The claim now follows from Lemma 6.9(ii). ♦

Remark 6.10. As far as explicit computations are concerned, evaluating the sum in Theorem 6.7 is often considerably easier than $\mathfrak{P}$-adic computations using Theorem 4.10 for two reasons. First, in contrast to the determination of the numbers $\#\bar{V}_{\boldsymbol{g}}^\tau(\mathfrak{O}/\mathfrak{P})$ (as K, hence $\mathfrak{O}$, varies) in Theorem 4.10, the computation of the Euler characteristics $\kappa_{\boldsymbol{g}}^{\mathrm{rel}}\big(\mathrm{New}(f^\tau)\big)_{f\in\boldsymbol{f}}$ in Theorem 6.7 is a purely mechanical process that can be performed by a computer. Second, while the rational functions occurring in Theorem 4.10 admit concise descriptions in terms of convex-geometric objects, when written as quotients of polynomials, they often become too unwieldy for further computations. Passing to the "reductions modulo $q-1$" of rational functions mitigates this to a large extent. It is an interesting and natural computational problem for future research to extend the practical scope of Theorem 4.10.

7 Applications

We now apply the techniques for computing $\mathfrak{P}$-adic integrals and topological zeta functions developed in the present article to some specific examples. Our primary focus will be on the case of topological subalgebra zeta functions. However, to illustrate Theorem 4.10, we also work through a known $\mathfrak{P}$-adic example.

7.1 Example: $\mathfrak{sl}_2(\mathbf{Z})$ – revisited

Let $\mathfrak{gl}_n(\mathbf{Z})$ be the Lie ring of all integral $n \times n$ matrices with the usual Lie bracket $[A, B] = AB - BA$. Let $\mathfrak{sl}_n(\mathbf{Z})$ be the Lie subring of $\mathfrak{gl}_n(\mathbf{Z})$ consisting of traceless matrices. We now illustrate the key ingredients of Theorem 4.10 in the context of the well-studied example $\mathfrak{sl}_2(\mathbf{Z})$.

History. The subalgebra zeta function of $\mathfrak{sl}_2(\mathbf{Z}_p)$ was first recorded (for odd p) by du Sautoy [19] based, among other things, on elaborate computations carried out by Ilani [30]. For $p \neq 2$, we have

$$\zeta_{\mathfrak{sl}_2(\mathbf{Z}_p)} = \frac{\zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-1)}{\zeta_p(3s-1)}, \tag{7.1}$$

where $\zeta_p(s) = 1/(1-p^{-s})$. Later, du Sautoy and Taylor [23] confirmed (7.1) by different means and also provided the missing case $p = 2$ (which differs from (7.1)). Their approach proceeds by computing an associated p-adic integral (see Theorem 2.6) via a manual resolution of singularities, obtained as a sequence of blow-ups with judiciously chosen centres. This geometrically-minded computation was later reinterpreted in a motivic setting by du Sautoy and Loeser, allowing them to deduce that

$$\zeta_{\mathfrak{sl}_2(\mathbf{Z}),\mathrm{top}}(\boldsymbol{s}) = \frac{3\boldsymbol{s}-1}{2(2\boldsymbol{s}-1)(\boldsymbol{s}-1)^2\boldsymbol{s}}, \tag{7.2}$$

see [21, §9.3] and note that we already corrected the shift present in [21], see Remark 5.18. For yet another p-adic approach, Klopsch and Voll [35] gave an explicit formula for $\zeta_{\mathcal{L}}(s)$, where $\mathcal{L}$ is a Lie $\mathbf{Z}_p$-algebra which is free of rank 3 as a $\mathbf{Z}_p$-module, in terms of Igusa's local zeta function associated with a ternary quadratic form attached to $\mathcal{L}$.

Setting up the integral. We now explain how the p-adic integral in [23] can, for odd p, be computed using our method. Let K be a finite extension of $\mathbf{Q}_p$. As before, we follow the conventions explained in Notation 2.5. First, by applying Theorem 2.6 and Remark 2.7(ii) with respect to the $\mathbf{Z}$-basis $\left(\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]\right)$ of $\mathfrak{sl}_2(\mathbf{Z})$, as stated in [23, §2] (for $K = \mathbf{Q}_p$), we obtain

$$\zeta_{\mathfrak{sl}_2(\mathfrak{O})}(s) = (1-q^{-1})^{-3} \int_{V(\mathfrak{O})} |x_{11}|^{s-1}|x_{22}|^{s-2}|x_{33}|^{s-3}\,\mathrm{d}\mu(\boldsymbol{x}), \tag{7.3}$$

where

$$V(\mathfrak{O}) = \left\{\boldsymbol{x} = \left[\begin{smallmatrix}x_{11}&x_{12}&x_{13}\\&x_{22}&x_{23}\\&&x_{33}\end{smallmatrix}\right] \in \mathrm{Tr}_3(\mathfrak{O}) \cap \mathbf{T}^6(K) : \|4x_{12}x_{22}^{-1}x_{23},\ 4x_{12}x_{22}^{-1}x_{33},\ f(\boldsymbol{x})\| \leqslant 1\right\},$$

$f(\boldsymbol{X}) = X_{11}X_{22}X_{33}^{-1} + 4X_{13}X_{23}X_{33}^{-1} - 4X_{12}X_{22}^{-1}X_{23}^{2}X_{33}^{-1}$, and we identify $\mathrm{Tr}_3(K) \approx K^6$ via $\boldsymbol{x} \mapsto (x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{33})$. We henceforth assume that $p \neq 2$. Let

$$\mathcal{C}_0 = \left\{(\omega_{11}, \omega_{12}, \dots, \omega_{33}) \in \mathbf{R}^6_{\geqslant 0} : \omega_{12} - \omega_{22} + \omega_{23} \geqslant 0,\ \omega_{12} - \omega_{22} + \omega_{33} \geqslant 0\right\}$$

so that $V(\mathfrak{O}) = \{\boldsymbol{x} \in \mathbf{T}^6(K) : \nu(\boldsymbol{x}) \in \mathcal{C}_0, |f(\boldsymbol{x})| \leqslant 1\}$. In order to keep technicalities in our exposition at a minimum and in view of the comparatively simple shape of the integral in (7.3), in the following, we will directly carry out the relevant steps from the proof of Theorem 4.10. In particular, we apply the relevant specialisations from Remark 4.12 right from the start and we also describe all rational functions in terms of generating functions of half-open cones instead of invoking the language of §3.3.

Breaking up the integral. The Newton polytope $\Delta := \mathrm{New}(f) \subset \mathbf{R}^6$ is a triangle and one checks that f is globally non-degenerate over $\mathbf{Q}$. The same is true of the reduction of f modulo every prime—note, however, that 2 is an exceptional prime for f in the sense that the Newton polytope of the reduction of f modulo 2 differs from that of f.

As $(1,\dots,1)$ is an interior point of $\mathcal{C}_0$ and f is homogeneous, each of the 7 faces of Δ is $\mathcal{C}_0$-visible by Remark 4.3(iv). Let us write $\boldsymbol{a} := X_{11}X_{22}X_{33}^{-1}$, $\boldsymbol{b} := 4X_{13}X_{23}X_{33}^{-1}$, and $\boldsymbol{c} := -4X_{12}X_{22}^{-1}X_{23}^{2}X_{33}^{-1}$ for the terms of f. We further use the same symbols to denote the corresponding vertices of Δ and let $\boldsymbol{ab}$, $\boldsymbol{ac}$, and $\boldsymbol{bc}$ denote the edges of Δ suggested by the notation. Note that in the present situation, each non-zero "sub-polynomial" of f gives rise to a face of Δ. As in the proof of Proposition 3.8, for a face $\tau \subset \Delta$, define $\mathcal{C}_0^\tau := \mathcal{C}_0 \cap \mathrm{N}_\tau(\Delta)$. Let $V^\tau(\mathfrak{O}) := \{\boldsymbol{x} \in \mathbf{T}^6(K) : \nu(\boldsymbol{x}) \in \mathcal{C}_0^\tau, |f(\boldsymbol{x})| \leqslant 1\}$ and $I^\tau(s) := \int_{V^\tau(\mathfrak{O})} |x_{11}|^{s-1}|x_{22}|^{s-2}|x_{33}|^{s-3}\,\mathrm{d}\mu(\boldsymbol{x})$ so that $(1-q^{-1})^3\zeta_{\mathfrak{sl}_2(\mathfrak{O})}(s) = \sum_\tau I^\tau(s)$. Write $\alpha,\beta,\gamma \in \mathbf{Z}^6$ for the exponent vectors of the monomials in $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively. We thus, for example, have $\mathcal{C}_0^{\boldsymbol{ab}} = \{\omega \in \mathcal{C}_0 : \langle\alpha,\omega\rangle = \langle\beta,\omega\rangle < \langle\gamma,\omega\rangle\}$.

Computing the pieces: vertices. We now describe the computation of $I^{\boldsymbol{a}}(s)$. The other two vertices $\boldsymbol{b}$ and $\boldsymbol{c}$ of Δ can be treated in the same way. Thus, if $\omega \in \mathcal{C}_0^{\boldsymbol{a}} \cap \mathbf{Z}^6$ and $\boldsymbol{x} \in \mathbf{T}^6(K)$ with $\nu(\boldsymbol{x}) = \omega$, then $\nu(f(\boldsymbol{x})) = \langle\alpha,\omega\rangle$ as $\mathrm{in}_\omega(f) = \boldsymbol{a}$ does not vanish on $\mathbf{T}^6(\mathfrak{O}/\mathfrak{P})$. Letting $(\,\cdot\,)^*$ denote dual cones and writing $|\cdot|$ for generating functions of half-open cones (see §3.1), as in the proof of Proposition 3.9, we find that

$$\begin{aligned} I^{\boldsymbol{a}}(s) &= \int\limits_{\{\boldsymbol{x}\in\mathbf{T}^6(K):\nu(\boldsymbol{x})\in\mathcal{C}_0^{\boldsymbol{a}}\cap\{\alpha\}^*\}} |x_{11}|^{s-1}|x_{22}|^{s-2}|x_{33}|^{s-3}\,\mathrm{d}\mu(\boldsymbol{x}) \\ &= (1-q^{-1})^6\,\underbrace{|\mathcal{C}_0^{\boldsymbol{a}} \cap \{\boldsymbol{a}\}^*|\left(\begin{bmatrix} q^{-s} & q^{-1} & q^{-1} \\ & q^{1-s} & q^{-1} \\ & & q^{2-s}\end{bmatrix}\right)}_{=:Z^{\boldsymbol{a}}(s)}, \end{aligned} \tag{7.4}$$

where the substitution is to be understood in the natural way—that is, within $|\mathcal{C}_0^{\boldsymbol{a}} \cap \{\boldsymbol{a}\}^*|(\xi_{11},\dots,\xi_{33})$, replace ξ_{ij} by the entry in position (i,j) of the matrix in (7.4).

Computing the pieces: edges. We now consider the computation of $I^{\boldsymbol{ab}}(s)$, the cases of the other two edges $\boldsymbol{ac}$ and $\boldsymbol{bc}$ of Δ being essentially identical. Let $\omega \in \mathcal{C}_0^{\boldsymbol{ab}} \cap \mathbf{Z}^6$ and $\boldsymbol{u} \in \mathbf{T}^6(\mathfrak{O})$ be arbitrary; note that $\langle\alpha,\omega\rangle = \langle\beta,\omega\rangle$. Using §4.1, we see that $f(\pi^\omega\boldsymbol{u}) = \pi^{\langle\alpha,\omega\rangle}\cdot\big((\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{u}) + \mathcal{O}(\pi)\big)$. Therefore, if $(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{u}) \not\equiv 0 \bmod \mathfrak{P}$, then $\nu(f(\pi^\omega\boldsymbol{x})) = \langle\alpha,\omega\rangle$ for all $\boldsymbol{x} \in \boldsymbol{u} + \mathfrak{P}^6$. Suppose that $(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{u}) \equiv 0 \bmod \mathfrak{P}$. Then non-degeneracy of f and Hensel's lemma allow us to to replace $\boldsymbol{u}$ by a possibly different element of $\boldsymbol{u}+\mathfrak{P}^6$ in such a way that the congruence $(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{u}) \equiv 0 \bmod \mathfrak{P}$ becomes an equality $f(\pi^\omega\boldsymbol{u}) = 0$. Next, after a suitable local change of coordinates, we may even assume that $f(\pi^\omega\boldsymbol{x}) = \pi^{\langle\alpha,\omega\rangle}(x_1 - u_1)$ for all $\boldsymbol{x} \in \boldsymbol{u}+\mathfrak{P}^6$, see the proof of Theorem 4.10 for details. The number of solutions of $(\boldsymbol{a}+\boldsymbol{b})(\bar{\boldsymbol{u}}) \equiv 0 \bmod \mathfrak{P}$ for $\bar{\boldsymbol{u}} \in \mathbf{T}^6(\mathfrak{O}/\mathfrak{P})$ is $(q-1)^5$, as we can e.g. solve for $\bar{u}_{11}$. We conclude that

$$I^{\boldsymbol{ab}}(s) = \frac{(q-1)^6-(q-1)^5}{q^6}\cdot\underbrace{|\mathcal{C}_0^{\boldsymbol{ab}} \cap \{\boldsymbol{a}\}^*|(\dots)}_{=:Z^{\boldsymbol{ab}}_{\mathrm{off}}(s)} + \frac{(q-1)^{5+1}}{q^6}\cdot\underbrace{|(\mathcal{C}_0^{\boldsymbol{ab}}\times\mathbf{R}_{>0}) \cap \{(\alpha,1)\}^*|(\dots)}_{=:Z^{\boldsymbol{ab}}_{\mathrm{on}}(s)},$$

where the substitutions are as in the case of $I^{\boldsymbol{a}}(s)$ above, except that the additional variable in the second summand is also replaced by q^{-1}. The "on/off"-types of rational functions, indicating the vanishing or non-vanishing of the initial form $\boldsymbol{a}+\boldsymbol{b}$ of f, correspond to the two subsets $\boldsymbol{g} \subset \{f\}$ in Theorem 4.10, see below.

Computing the pieces: the triangle. The computation for the triangle $\tau = \Delta$ is very similar to those for the edges. It is easy to see that the number of points $\bar{\boldsymbol{u}} \in \mathbf{T}^6(\mathfrak{O}/\mathfrak{P})$ with $f(\bar{\boldsymbol{u}}) \equiv 0 \bmod \mathfrak{P}$ is $(q-1)^4(q-2)$.

Summing up. In summary, we obtain rational functions $Z^{\tau}_{\mathrm{on/off}}(s) \in \mathbf{Q}(q, q^{-s})$ with

$$\begin{aligned}(1-q^{-1})^3\zeta_{\mathfrak{sl}_2(\mathfrak{O})}(s) =&(1-q^{-1})^6 \cdot (Z^{\boldsymbol{a}}(s) + Z^{\boldsymbol{b}}(s) + Z^{\boldsymbol{c}}(s) + Z^{\boldsymbol{ab}}_{\mathrm{on}}(s) + Z^{\boldsymbol{ac}}_{\mathrm{on}}(s) + Z^{\boldsymbol{bc}}_{\mathrm{on}}(s))+\\ &(1-q^{-1})^5(1-2q^{-1}) \cdot (Z^{\boldsymbol{ab}}_{\mathrm{off}}(s) + Z^{\boldsymbol{ac}}_{\mathrm{off}}(s) + Z^{\boldsymbol{bc}}_{\mathrm{off}}(s))+\\ &(1-q^{-1})^5(1-2q^{-1})Z^{\Delta}_{\mathrm{on}}(s) + (1-q^{-1})^4(1-3q^{-1}+3q^{-2})Z^{\Delta}_{\mathrm{off}}(s).\end{aligned} \tag{7.5}$$

All that remains to be done to finish our computation of $\zeta_{\mathfrak{sl}_2(\mathfrak{O})}(s)$ is to explicitly evaluate the right-hand side of (7.5). In order to do that, we only need to compute generating functions of certain half-open cones which can e.g. be determined using the computer program `LattE` [40]. Carrying out this computation, after substantial cancellations, we find that

$$\zeta_{\mathfrak{sl}_2(\mathfrak{O})}(s) = \frac{1-q^{1-3s}}{(1-q^{1-2s})(1-q^{2-2s})(1-q^{1-s})(1-q^{-s})}, \tag{7.6}$$

which of course agrees with (7.1) for $\mathfrak{O} = \mathbf{Z}_p$; recall that we assumed that $\mathfrak{P} \cap \mathbf{Z} \neq \langle 2 \rangle$.

Remark 4.12 allows us to compare (7.5) and the formula in Theorem 4.10. Since the integrand in (7.3) consists of monomials, the only subsets $\boldsymbol{g} \subset \{f, X_{11}, X_{22}, X_{33}\}$ that can possibly give rise to a non-zero number $\#\bar{V}^{\tau}_{\boldsymbol{g}}(\mathfrak{O}/\mathfrak{P})$ are precisely the subsets of $\{f\}$—the "on/off-types" of rational functions correspond to those. Note that if τ is a vertex, then the corresponding initial form of f does not vanish on the torus. This explains the absence of an "on/off"-distinction for vertices.

Having carried out the above computation for arbitrary $\mathfrak{O}$ with $\mathfrak{P} \cap \mathbf{Z} \neq \langle 2 \rangle$, it is a straightforward matter to confirm the associated topological zeta function given in (7.2). For larger examples, $\mathfrak{P}$-adic computations often become considerably more involved than their topological counterparts (see Remark 6.10). From now on, we will focus exclusively on the topological case.

7.2 Example: from the Heisenberg Lie ring to quadratic forms

Let k be a number field with ring of integers $\mathfrak{o}$. Let M be a free $\mathfrak{o}$-module of finite rank endowed with a bilinear form $\beta\colon M \times M \to \mathfrak{o}$. We assume that β is non-degenerate over k. Define an $\mathfrak{o}$-algebra $\mathcal{B}(\beta)$ with underlying $\mathfrak{o}$-module $M \oplus \mathfrak{o}$ and multiplication $(\boldsymbol{x}, a)(\boldsymbol{y}, b) = (0, \beta(\boldsymbol{x}, \boldsymbol{y}))$. Such algebras were considered in [22, §3.3] in the context of a different type of zeta function.

Example 7.1. If $k = \mathbf{Q}$, $M = \mathbf{Z}^{2m}$ and β is the standard symplectic form $\beta(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} \cdot \left[\begin{smallmatrix} 0_m & 1_m \\ -1_m & 0_m \end{smallmatrix}\right] \cdot \boldsymbol{y}^\top$, then $\mathcal{B}(\beta)$ is a central product $\mathcal{B}(\beta) \approx \mathfrak{h} \curlyvee \cdots \curlyvee \mathfrak{h}$ of m copies of the Heisenberg Lie ring $\mathfrak{h} = \left[\begin{smallmatrix} 0 & \mathbf{Z} & \mathbf{Z} \\ & 0 & \mathbf{Z} \\ & & 0 \end{smallmatrix}\right] \leqslant \mathfrak{gl}_3(\mathbf{Z})$.

Remark 7.2. Using the conventions in Notation 2.5, for a non-Archimedean local field $K \supset k$, the subalgebra zeta function $\zeta_{\mathcal{B}(\beta)\otimes_{\mathfrak{o}}\mathfrak{O}}(s)$ is closely related to arithmetic properties of the bilinear forms induced by β on submodules of $M_{\mathfrak{O}}$. Namely, for each $\mathfrak{O}$-submodule $\Lambda \leqslant M_{\mathfrak{O}}$ of finite index, define $m(\Lambda) := \min(\nu(\beta(\boldsymbol{x}, \boldsymbol{y})) : \boldsymbol{x}, \boldsymbol{y} \in \Lambda) < \infty$ and consider the bivariate zeta function

$$\mathsf{Z}^\beta(s_1, s_2) = \sum_\Lambda |M_{\mathfrak{O}} : \Lambda|^{-s_1} q^{-m(\Lambda)s_2}.$$

Then it is easy to see that $(q^{d-s} - 1)\zeta_{\mathcal{B}(\beta)\otimes_{\mathfrak{o}}\mathfrak{O}}(s) = q^{d-s}\mathsf{Z}^\beta(s, s-d) - \zeta_{(\mathfrak{O}^d,0)}(s)$, where $(\mathfrak{O}^d, 0)$ denotes $\mathfrak{O}^d$ endowed with the zero multiplication and $d = \mathrm{rk}_{\mathfrak{o}}(M)$.

Heisenberg Lie rings. Let K be a p-adic field with associated objects as in Notation 2.5. The subalgebra zeta function of $\mathfrak{h}_{\mathfrak{O}}$ was among the very first examples computed, see [27, Prop. 8.1]; their formula for $\mathfrak{O} = \mathbf{Z}_p$ and its proof carry over to the present case. The topological result, $\zeta_{\mathfrak{h},\mathrm{top}}(\boldsymbol{s}) = \frac{3}{2(2\boldsymbol{s}-3)(\boldsymbol{s}-1)\boldsymbol{s}}$, has been recorded in [21, §9.2] (up to the shift explained in Remark 5.18). The subalgebra zeta function of $\mathfrak{h}_{\mathbf{Z}_p} \curlyvee \mathfrak{h}_{\mathbf{Z}_p}$ was first computed by Woodward, see [55, p. 57] for the result.

We now sketch the computation of $\zeta_{\mathfrak{h}\curlyvee\mathfrak{h},\mathrm{top}}(\boldsymbol{s})$ using our method. First, we choose a $\mathbf{Z}$-basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_5)$ of $\mathfrak{h} \curlyvee \mathfrak{h}$ which satisfies (a) $[\boldsymbol{e}_1, \boldsymbol{e}_3] = [\boldsymbol{e}_2, \boldsymbol{e}_4] = \boldsymbol{e}_5$ and (b) all other commutators of basis elements not implied by anti-symmetry are zero. Next, Theorem 2.6 and Remark 2.7(ii) provide us with a description of $\zeta_{\mathfrak{h}_{\mathfrak{O}}\curlyvee\mathfrak{h}_{\mathfrak{O}}}(s)$ in terms of a $\mathfrak{P}$-adic integral. As illustrated in the case of $\mathfrak{sl}_2(\mathbf{Z})$ in §7.1, by discarding finitely many primes, we can always move monomial divisibility conditions from the domain of integration into the ambient cone $\mathcal{C}_0$. Moreover, since the integrand in Theorem 2.6 is always comprised of monomials, it is without relevance for questions of non-degeneracy. In the present context, a simple computation reveals the essential ingredient of the $\mathfrak{P}$-adic integral in Theorem 2.6 to be the condition

$$\|\underbrace{x_1x_{10}x_{15}^{-1} + x_2x_{11}x_{15}^{-1}}_{=:f_1(\boldsymbol{x})}, \quad \underbrace{x_4x_6x_{15}^{-1} - x_1x_7x_{15}^{-1} - x_2x_8x_{15}^{-1}}_{=:f_2(\boldsymbol{x})}\| \leqslant 1,$$

where $\boldsymbol{x} \in \mathbf{T}^{15}(K) \cap \mathfrak{O}^{15}$ and we identified $\mathrm{Tr}_5(K) \approx K^{15}$. One can check that $\mathrm{New}(f_1f_2)$ is a 3-dimensional polytope with 6 vertices and a total of 21 faces. Most importantly (f_1, f_2) is globally non-degenerate. By following through our procedure (Theorem 6.7) with the help of a computer, we find that

$$\zeta_{\mathfrak{h}\curlyvee\mathfrak{h},\mathrm{top}}(\boldsymbol{s}) = \frac{17\boldsymbol{s} - 21}{3(3\boldsymbol{s} - 4)(3\boldsymbol{s} - 7)(\boldsymbol{s} - 3)(\boldsymbol{s} - 2)(\boldsymbol{s} - 1)\boldsymbol{s}},$$

which is exactly what one would expect given Woodward's p-adic formula and the "informal approach" from the introduction.

While we shall not concern ourselves with it in any detail, the complexity of the integral under consideration here is sufficiently low to allow for computer-assisted $\mathfrak{P}$-adic computations. In order to adapt our computations for $\mathfrak{sl}_2(\mathfrak{O})$ in §7.1 to the case of $\mathfrak{h}_{\mathfrak{O}} \curlyvee \mathfrak{h}_{\mathfrak{O}}$, one would replace the "on/off"-distinction used there by pairs (on/off, on/off) indicating exactly which of the initial forms of f_1 and f_2 vanish at a given point of $\mathbf{T}^{15}(\mathfrak{O}/\mathfrak{P})$. This suggests a total number of $4 \cdot 21$ cases but, as in §7.1, this number can be reduced slightly because certain subvarieties of tori are a priori known to be empty.

Commutative variants of Heisenberg rings. Let $\mathfrak{ch}_d$ be the $\mathbf{Z}$-algebra $\mathcal{B}(\beta_d)$, where β_d is the standard inner product $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \sum_{i=1}^{d} x_i y_i$ on $\mathbf{Z}^d$. In view of Example 7.1, we regard the rings $\mathfrak{ch}_d$ as (non-unital) commutative analogues of the central products of Heisenberg Lie rings considered above. We will now describe how we computed

$$\begin{aligned} \zeta_{\mathfrak{ch}_2,\mathrm{top}}(\boldsymbol{s}) &= \frac{3\boldsymbol{s} - 2}{4(\boldsymbol{s}-1)^3 \boldsymbol{s}}, \\ \zeta_{\mathfrak{ch}_3,\mathrm{top}}(\boldsymbol{s}) &= \frac{14}{(5\boldsymbol{s}-6)(5\boldsymbol{s}-7)(\boldsymbol{s}-2)\boldsymbol{s}}, \text{ and} \\ \zeta_{\mathfrak{ch}_4,\mathrm{top}}(\boldsymbol{s}) &= \frac{17\boldsymbol{s} - 25}{(3\boldsymbol{s}-4)(3\boldsymbol{s}-5)^2(\boldsymbol{s}-3)(\boldsymbol{s}-2)\boldsymbol{s}} \end{aligned}$$

using Theorem 6.7 and some elementary insight into the structure of $\mathfrak{ch}_d$. To the author's knowledge, these topological zeta functions have not been previously computed, nor are their $\mathfrak{P}$-adic versions known.

Write $\mathfrak{ch}_d = \mathbf{Z}\boldsymbol{e}_1 \oplus \cdots \oplus \mathbf{Z}\boldsymbol{e}_d \oplus \mathbf{Z}\boldsymbol{v}$, where $\boldsymbol{e}_i^2 = \boldsymbol{v}$ and all other products of basis vectors are trivial. We again follow the conventions in Notation 2.5. By using Theorem 2.6 and Remark 2.7(ii) relative to the $\mathbf{Z}$-basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{v})$ of $\mathfrak{ch}_2$, we express $\zeta_{\mathfrak{ch}_2(\mathfrak{O})}(s)$ in terms of a $\mathfrak{P}$-adic integral. If we identify $\mathrm{Tr}_3(K) \approx K^6$ as above, the domain of integration consists of those $\boldsymbol{x} \in \mathfrak{O}^6 \cap \mathbf{T}^6(K)$ with $\|x_1^2 x_6^{-1} + x_2^2 x_6^{-1}, x_4^2 x_6^{-1}, x_2 x_4 x_6^{-1}\| \leqslant 1$. As before, monomials do not affect questions of degeneracy. Moreover, binomials are always globally non-degenerate so the computation of $\zeta_{\mathfrak{ch}_2,\mathrm{top}}(\boldsymbol{s})$ becomes an essentially trivial matter. In contrast, the $\mathbf{Z}$-basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{v})$ of $\mathfrak{ch}_3$ gives rise to a degenerate system of polynomials.

In order to nonetheless be able to compute $\zeta_{\mathfrak{ch}_3,\mathrm{top}}(\boldsymbol{s})$ and $\zeta_{\mathfrak{ch}_4,\mathrm{top}}(\boldsymbol{s})$, we use some elementary facts about quadratic forms to construct bilinear forms γ_d with $\zeta_{\mathfrak{ch}_d,\mathrm{top}}(\boldsymbol{s}) = \zeta_{\mathcal{B}(\gamma_d),\mathrm{top}}(\boldsymbol{s})$, where $\mathcal{B}(\gamma_d)$ is now amenable to our methods for $d = 3$ and $d = 4$. Thus, write $[a_1, \dots, a_r]$ for the form $(x_1, \dots, x_r) \mapsto a_1 x_1^2 + \cdots + a_r x_r^2$, where $r \geqslant 0$. Let $h(x_1, x_2) = x_1 x_2$ be the standard hyperbolic form and write $h^{\perp a} = h \perp \cdots \perp h$ (a copies). Define a quadratic form f_d in d variables as follows: for $d = 4a + b$ with $0 \leqslant b < 4$, let $f_d = h^{\perp 2a} \perp g_b$, where $g_0 = [\,]$, $g_1 = [1]$, $g_2 = [1, 1]$, and $g_3 = h \perp [-1]$.

Lemma. *If $p \neq 2$, then $(x_1, \dots, x_d) \mapsto x_1^2 + \cdots + x_d^2$ and f_d are equivalent over $\mathbf{Z}_p[\sqrt{2}]$.*

Proof. Let $\sim$ signify equivalence over $\mathbf{Z}_p[\sqrt{2}]$. By Hensel's lemma, there exist $x, y \in \mathbf{Z}_p$ with $x^2 + y^2 = -1$. Let $A = \left[\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right]$ and $B = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Then $AA^\top = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $B\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]B^\top = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Hence, $[1,1,1] \sim [-1,-1,1] \sim g_3$ and $[1,1,1,1] \sim h \perp h$. ♦

By quadratic reciprocity, if $p \neq 2$, then $\mathbf{Z}_p$ contains $\sqrt{2}$ if and only if $p \equiv \pm 1 \bmod 8$. Let γ_d be the symmetric bilinear form on $\mathbf{Z}^d$ with $\gamma_d(\boldsymbol{x}, \boldsymbol{x}) = f_d(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbf{Z}^d$.

Corollary. *If $p \neq 2$, then $\mathfrak{ch}_d \otimes \mathbf{Z}_p[\sqrt{2}] \approx_{\mathbf{Z}_p[\sqrt{2}]} B(\gamma_d) \otimes \mathbf{Z}_p[\sqrt{2}]$. Hence, $\zeta_{\mathfrak{ch}_d,\mathrm{top}}(\boldsymbol{s}) = \zeta_{\mathcal{B}(\gamma_d),\mathrm{top}}(\boldsymbol{s})$.* ♦

In contrast to $\mathfrak{ch}_3$ and $\mathfrak{ch}_4$, our methods do apply to $\mathcal{B}(\gamma_3)$ and $\mathcal{B}(\gamma_4)$. By choosing bases similar to the cases of central products of Heisenberg Lie rings, disregarding monomials as before, the computation of $\zeta_{\mathcal{B}(\gamma_3),\mathrm{top}}(\boldsymbol{s})$ involves a non-degenerate pair of binomials, while for $\zeta_{\mathcal{B}(\gamma_4),\mathrm{top}}(\boldsymbol{s})$, we face a non-degenerate triple consisting of a trinomial and two binomials. The situation for $\mathcal{B}(\gamma_5)$ is more complicated and, in particular, degenerate.

7.3 Outlook: conquering degeneracy

The examples provided in §§7.1–7.2 show that the techniques developed in the present article can be used to compute interesting examples of $\mathfrak{P}$-adic and topological subalgebra (and, similarly, submodule) zeta functions. Unfortunately, for typical examples of algebras and modules of interest, more often than not, our non-degeneracy assumptions tend to be violated. In [44], we describe a refined form of the method developed here and we design additional techniques to overcome certain forms of degeneracy. These improvements and extensions considerably extend the scope of our method. To provide some further motivation for the conjectures given in §8, we now provide some examples computed with the help of [44].

Example: $\mathfrak{gl}_2(\mathbf{Z})$

To the author's knowledge, up until now, $\mathfrak{sl}_2(\mathbf{Z})$ remained the sole example of an insoluble Lie ring whose topological or $\mathfrak{P}$-adic subalgebra zeta function has been computed. Using [44], we can add the topological subalgebra zeta function of $\mathfrak{gl}_2(\mathbf{Z})$ to the list:

$$\zeta_{\mathfrak{gl}_2(\mathbf{Z}),\mathrm{top}}(\boldsymbol{s}) = \frac{27\boldsymbol{s} - 14}{6(6\boldsymbol{s} - 7)(\boldsymbol{s} - 1)^3\boldsymbol{s}}. \tag{7.7}$$

Example: nilpotent groups of Hirsch length at most 5

Let Fil_4 be the nilpotent Lie ring with $\mathbf{Z}$-basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_5)$ such that $[\boldsymbol{e}_1, \boldsymbol{e}_2] = \boldsymbol{e}_3$, $[\boldsymbol{e}_1, \boldsymbol{e}_3] = \boldsymbol{e}_4$, $[\boldsymbol{e}_1, \boldsymbol{e}_4] = \boldsymbol{e}_5$, and $[\boldsymbol{e}_2, \boldsymbol{e}_3] = \boldsymbol{e}_5$, where the remaining commutators $[\boldsymbol{e}_i, \boldsymbol{e}_j]$ not implied by anti-commutativity are taken to be zero. The ideal zeta function of $\mathrm{Fil}_4 \otimes \mathbf{Z}_p$ was computed by Woodward [24, Thm 2.39]. "Despite repeated efforts" [24, p. 54] of his to compute them, the local subalgebra zeta functions of Fil_4 remain unknown. Using the methods from the present article and the techniques from [44], we find that

$$\begin{aligned}\zeta_{\mathrm{Fil}_4,\mathrm{top}}(\boldsymbol{s}) = \big(&392031360\boldsymbol{s}^9 - 5741480808\boldsymbol{s}^8 + 37286908278\boldsymbol{s}^7 - \\ &140917681751\boldsymbol{s}^6 + 341501393670\boldsymbol{s}^5 - 550262853249\boldsymbol{s}^4 + \\ &589429290044\boldsymbol{s}^3 - 404678115300\boldsymbol{s}^2 + 161557332768\boldsymbol{s} - \\ &28569052512\big)\big/\big(3(15\boldsymbol{s}-26)(7\boldsymbol{s}-12)(7\boldsymbol{s}-13)(6\boldsymbol{s}-11)^3 \\ &\quad (5\boldsymbol{s}-8)(5\boldsymbol{s}-9)(4\boldsymbol{s}-7)^2(3\boldsymbol{s}-4)(2\boldsymbol{s}-3)(\boldsymbol{s}-1)\boldsymbol{s}\big). \end{aligned} \tag{7.8}$$

Our conjectures in §8 suggest that the numbers in (7.8) are far from random.

As we will now explain, our computation of $\zeta_{\mathrm{Fil}_4,\mathrm{top}}(\boldsymbol{s})$ concludes the classification of topological subgroup zeta functions (see the end of §5.4) of all finitely generated torsion-free nilpotent groups G with $\mathrm{h}(G) \leqslant 5$, where $\mathrm{h}(G)$ denotes the Hirsch length of G. Recall that the topological zeta function $\zeta_{G,\mathrm{top}}(\boldsymbol{s})$ of such a group G only depends on the $\mathbf{C}$-isomorphism type of $\mathfrak{L}(G) \otimes_{\mathbf{Q}} \mathbf{C}$, where $\mathfrak{L}(G)$ is the $\mathrm{h}(G)$-dimensional Lie algebra over $\mathbf{Q}$ associated with G under the Mal'cev correspondence. It is well-known that, up to $\mathbf{C}$-isomorphism, there are precisely 16 non-trivial nilpotent Lie algebras of dimension at most 5 over $\mathbf{C}$ (see e.g. [12, §4]) and all these Lie algebras admit $\mathbf{Z}$-forms. With the sole exception of $\mathrm{Fil}_4 \otimes \mathbf{C}$, each of these Lie algebras over $\mathbf{C}$ admits a $\mathbf{Z}$-form whose p-adic subalgebra zeta functions have been recorded in [24]. We can use the techniques from [44] to compute the associated topological zeta functions of these Lie rings. As expected, it turns out that in all cases, the results coincide with those obtained by naively applying the non-rigorous "informal approach" from the introduction to the p-adic formulae in [24]. In this sense, $\zeta_{\mathrm{Fil}_4,\mathrm{top}}(\boldsymbol{s})$ was indeed the only missing case which we now managed to compute without resorting to a computation of the associated p-adic zeta functions.

Example: submodules for unipotent representations

The study of submodule zeta functions arising from orders in semisimple associative $\mathbf{Q}$-algebras was initiated by Solomon [47]. In particular, he gave explicit formulae, valid for almost all primes, for the local factors in terms of the Wedderburn decomposition of the associated $\mathbf{Q}$-algebra. At the other end of the spectrum lie ideal zeta functions of nilpotent Lie rings (cf. Remark 2.2(ii)), a great number of which have been recorded in [24, Ch. 2]. We now consider topological zeta functions arising from the enumeration of submodules of $\mathbf{Z}^d$ under the action of the unipotent group

$$\mathrm{U}_d(\mathbf{Z}) = \begin{bmatrix} 1 & \mathbf{Z} & \cdots & \mathbf{Z} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{Z} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \leqslant \mathrm{GL}_d(\mathbf{Z})$$

for $d \leqslant 5$. That is, using the language of §2, we consider $\zeta_{\mathcal{U}_d(\mathbf{Z}) \curvearrowright \mathbf{Z}^d,\mathrm{top}}(\boldsymbol{s})$, where $\mathcal{U}_d(\mathbf{Z})$ is the associative $\mathbf{Z}$-subalgebra of $\mathrm{M}_d(\mathbf{Z})$ generated by $\mathrm{U}_d(\mathbf{Z})$. The author is not aware of any previous computations of these zeta functions or their $\mathfrak{P}$-adic versions.

It is a trivial matter to determine the first two examples

$$\zeta_{\mathcal{U}_2(\mathbf{Z})\curvearrowright\mathbf{Z}^2,\mathrm{top}}(\boldsymbol{s}) = \frac{1}{(2\boldsymbol{s}-1)\boldsymbol{s}}, \qquad \zeta_{\mathcal{U}_3(\mathbf{Z})\curvearrowright\mathbf{Z}^3,\mathrm{top}}(\boldsymbol{s}) = \frac{4\boldsymbol{s}-1}{2(3\boldsymbol{s}-1)(2\boldsymbol{s}-1)^2\boldsymbol{s}}. \tag{7.9}$$

For $\mathrm{U}_4(\mathbf{Z})$, a computation similar to the case of $\mathfrak{ch}_2$ in §7.2 shows that

$$\zeta_{\mathcal{U}_4(\mathbf{Z})\curvearrowright\mathbf{Z}^4,\mathrm{top}}(\boldsymbol{s}) = \frac{3360\boldsymbol{s}^5 - 5192\boldsymbol{s}^4 + 3139\boldsymbol{s}^3 - 930\boldsymbol{s}^2 + 136\boldsymbol{s} - 8}{8(7\boldsymbol{s}-3)(5\boldsymbol{s}-2)(4\boldsymbol{s}-1)(3\boldsymbol{s}-1)^2(2\boldsymbol{s}-1)^3\boldsymbol{s}}. \tag{7.10}$$

The computation for $\mathrm{U}_5(\mathbf{Z})$ is noticeably more involved. Using [44], we obtain

$$\begin{aligned}\zeta_{\mathcal{U}_5(\mathbf{Z})\curvearrowright\mathbf{Z}^5,\mathrm{top}}(\boldsymbol{s}) = \big(&435891456000\boldsymbol{s}^{14} - 1957609382400\boldsymbol{s}^{13} + 4053337786080\boldsymbol{s}^{12} - \\ &5128457632240\boldsymbol{s}^{11} + 4429884320966\boldsymbol{s}^{10} - 2763848527457\boldsymbol{s}^9 + \\ &1284781950540\boldsymbol{s}^8 - 452226036325\boldsymbol{s}^7 + 121188554644\boldsymbol{s}^6 - \\ &24624421916\boldsymbol{s}^5 + 3737984412\boldsymbol{s}^4 - 411498360\boldsymbol{s}^3 + 31087152\boldsymbol{s}^2 - \\ &1443744\boldsymbol{s} + 31104\big)\big/\big(36(13\boldsymbol{s}-6)(11\boldsymbol{s}-4)(10\boldsymbol{s}-3)(9\boldsymbol{s}-4)(8\boldsymbol{s}-3) \\ &(7\boldsymbol{s}-2)(7\boldsymbol{s}-3)(5\boldsymbol{s}-1)(5\boldsymbol{s}-2)^2(4\boldsymbol{s}-1)^2(3\boldsymbol{s}-1)^2(2\boldsymbol{s}-1)^4\boldsymbol{s}\big).\end{aligned} \tag{7.11}$$

8 Conjectures

Based on substantial experimental evidence gathered using the method described in this article and its extension in [44], we now state and discuss a series of conjectures on topological zeta functions of algebras and modules. We first explain our conjectures in the context of subalgebra zeta functions.

8.1 General conjectures

Let $\mathcal{A}$ be a possibly non-associative $\mathfrak{o}$-algebra which is free of rank d as an $\mathfrak{o}$-module, where $\mathfrak{o}$ is the ring of integers in a number field k. Experimental evidence (for $k = \mathbf{Q}$) suggests that the following three conjectures hold without further assumptions on $\mathcal{A}$.

Conjecture I. $\deg_{\boldsymbol{s}}(\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s})) = -d$.

This is in contrast to topological zeta functions of polynomials in the sense that for $f \in k[X_1,\dots,X_n]$, the degree of $\mathsf{Z}_{f,\mathrm{top}}(\boldsymbol{s})$ in $\boldsymbol{s}$ does not merely depend on n.

If true, Conjecture I has the following $\mathfrak{P}$-adic consequence which, to the author's knowledge, has not been recorded before. Suppose that there are numbers $a_i \in \mathbf{Z}$, $b_i \in \mathbf{Z}\setminus\{0\}$, and $\varepsilon_i = \pm 1$ ($i \in I$, for finite I) with the following property:

(♣) For every p-adic field $K \supset k$ with valuation ring $\mathfrak{O}$, unless p belongs to some finite exceptional set, we have

$$\zeta_{\mathcal{A}_{\mathfrak{O}}}(s) = \prod_{i\in I}(1 - q^{a_i - b_i s})^{\varepsilon_i},$$

where q is the residue field size of K.

For instance, (7.6) shows that (♣) is satisfied for $\mathcal{A} = \mathfrak{sl}_2(\mathbf{Z})$. Furthermore, various examples of local ideal zeta functions of nilpotent Lie rings given in [24, Ch. 2] satisfy the evident analogue of (♣) (at least when $K = \mathbf{Q}_p$, see §8.3 and cf. Remark 5.20).

Assuming that (♣) holds, one can deduce from Theorems 5.12 and 5.16 that $\sum_{i\in I} \varepsilon_i \geqslant -d$. Conjecture I now predicts that, in fact, the equality $\sum_{i\in I} \varepsilon_i = -d$ holds.

Conjecture II. *$\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s})$ has a pole at zero.*

Again, topological zeta functions of polynomials behave more erratically as the simple example of $\mathsf{Z}_{X^e,\mathrm{top}}(\boldsymbol{s}) = 1/(e\boldsymbol{s} + 1)$ shows. On the $\mathfrak{P}$-adic side, the meromorphic continuations of local subalgebra zeta functions seem to have poles at zero but the author is not aware of an explanation as to why this is so.

Conjecture III. *If $s \in \mathbf{C}$ satisfies $\zeta_{\mathcal{A},\mathrm{top}}(s) = 0$, then $0 < \mathrm{Re}(s) < d - 2$.*

The "$0 < \mathrm{Re}(s)$" part of Conjecture III would imply that the numerator of $\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s}) \in \mathbf{Q}(\boldsymbol{s})$ has alternating signs which is indeed the case for the examples given in §7. Regarding the upper bound of $d-2$ in Conjecture III, examples (e.g. $\mathfrak{ch}_2$ in §7.2) show that it cannot be lowered to $d - 3$.

8.2 The nilpotent case: behaviour at zero

Let $\mathcal{A}$ be as in §8.1. In particular, let d denote the rank of $\mathcal{A}$ as an $\mathfrak{o}$-module. If $\mathcal{A}$ is nilpotent, then experimental evidence suggests a considerably strengthened form of Conjecture II to be true. We shall henceforth assume that $\mathcal{A}$ is either Lie or (non-unital) associative so that there is no ambiguity as to what we mean by nilpotency of $\mathcal{A}$.

Conjecture IV (topological form)**.** *If $\mathcal{A}$ is nilpotent, then $\zeta_{\mathcal{A},\mathrm{top}}(\boldsymbol{s})$ has a simple pole at zero with residue $(-1)^{d-1}/(d-1)!$.*

For example, in the case of Fil_4 from §7.3, we have

$$\boldsymbol{s}\zeta_{\mathrm{Fil}_4,\mathrm{top}}(\boldsymbol{s})\big|_{\boldsymbol{s}=0} = \frac{-28569052512}{3(-26)(-12)(-13)(-11)^3(-8)(-9)(-7)^2(-4)(-3)(-1)} = 1/24.$$

Some assumption on $\mathcal{A}$ is certainly required in Conjecture IV. For example, while $\zeta_{\mathfrak{gl}_2(\mathbf{Z}),\mathrm{top}}(\boldsymbol{s})$ (see (7.7)) does have a simple pole at zero, the residue is $-1/3$ instead of $-1/6$. Also, zero need not be a simple pole in general. For example, consider the non-associative ring $\mathcal{R} = \mathbf{Z}\boldsymbol{e}_1 \oplus \mathbf{Z}\boldsymbol{e}_2 \oplus \mathbf{Z}\boldsymbol{e}_3$ with $\boldsymbol{e}_1^2 = \boldsymbol{e}_2$, $\boldsymbol{e}_2^2 = \boldsymbol{e}_3$, $\boldsymbol{e}_3^2 = \boldsymbol{e}_1$, and $\boldsymbol{e}_i\boldsymbol{e}_j = 0$ for $i \neq j$. Using the techniques from [44], we compute $\zeta_{\mathcal{R},\mathrm{top}}(\boldsymbol{s}) = \frac{(15\boldsymbol{s}-7)(5\boldsymbol{s}-1)}{3(7\boldsymbol{s}-3)(2\boldsymbol{s}-1)^2\boldsymbol{s}^2}$. Such examples can be found more easily for ideal zeta functions. For instance, if $(\mathbf{Z}^d, \cdot\,)$ denotes $\mathbf{Z}^d$ endowed with component-wise multiplication, then $\zeta^{\triangleleft}_{(\mathbf{Z}^d,\cdot),\mathrm{top}}(\boldsymbol{s}) = \boldsymbol{s}^{-d}$.

Conjecture IV might merely be the topological shadow of a $\mathfrak{P}$-adic conjecture which, to the author's knowledge, has not been previously noted:

Conjecture IV ($\mathfrak{P}$-adic form)**.** *Suppose that $\mathcal{A}$ is nilpotent. Let $K \supset k$ be a p-adic field with valuation ring $\mathfrak{O}$. Then $\zeta_{\mathcal{A}_{\mathfrak{O}}}(s)$ has a simple pole at zero and*

$$\left.\frac{\zeta_{\mathcal{A}_{\mathfrak{O}}}(s)}{\zeta_{(\mathfrak{O}^d,0)}(s)}\right|_{s=0} = 1.$$

Here, $(\mathfrak{O}^d, 0)$ denotes $\mathfrak{O}^d$ endowed with the zero multiplication—it is well-known that $\zeta_{(\mathfrak{O}^d,0)}(s) = \prod_{i=0}^{d-1}(1-q^{i-s})^{-1}$, where q is the residue field size of K. Also, we identified the function $\frac{\zeta_{\mathcal{A}_{\mathfrak{O}}}(s)}{\zeta_{(\mathfrak{O}^d,0)}(s)}$ (which is defined for $\mathrm{Re}(s) > d$, say) with its meromorphic continuation to the complex plane.

8.3 On topological submodule zeta functions

Some care has to be taken to adapt Conjectures I–IV to the case of submodule zeta functions, primarily because our definition of topological submodule zeta functions in §5.4 allows them to vanish. For example, if K is any p-adic field with valuation ring $\mathfrak{O}$ and residue field size q, then $\zeta_{\mathrm{M}_d(\mathfrak{O})\curvearrowright\mathfrak{O}^d}(s) = 1/(1-q^{-ds})$, where $\mathrm{M}_d(\mathfrak{O})$ acts on $\mathfrak{O}^d$ in the natural way; cf. [19, Prop. 4.1]. Hence, if $d > 1$, then $\zeta_{\mathrm{M}_d(\mathbf{Z})\curvearrowright\mathbf{Z}^d,\mathrm{top}}(\boldsymbol{s}) = 0$. We predict that Conjectures I–IV also hold for submodule zeta functions whenever they are non-zero. The nilpotency condition in Conjecture IV of course needs to be adapted: for a free $\mathfrak{o}$-module M of finite rank, we only consider nilpotent (associative) subalgebras $\mathcal{E}$ of $\mathrm{End}_{\mathfrak{o}}(M)$. We note that, although we do not have a proof of this, nilpotency of $\mathcal{E} \subset \mathrm{End}_{\mathfrak{o}}(M)$ seems to imply that $\zeta_{\mathcal{E}\curvearrowright M,\mathrm{top}}(\boldsymbol{s}) \neq 0$.

The submodule zeta functions arising from nilpotent subalgebras $\mathcal{E} \subset \mathrm{End}_{\mathfrak{o}}(M)$ include those enumerating ideals in nilpotent Lie rings. Indeed, if $\mathcal{L}$ is a nilpotent Lie algebra over $\mathfrak{o}$ which is free of finite rank as an $\mathfrak{o}$-module, then, by the standard proof of Engel's theorem (see e.g. [31, Ch. II, §§2–3]), $\mathrm{ad}(\mathcal{L})$ generates a nilpotent associative subalgebra $\mathcal{E}$ of $\mathrm{End}_{\mathfrak{o}}(\mathcal{L})$ and the $(\mathcal{E}+\mathfrak{o}1_{\mathcal{L}})$-submodules of $\mathcal{L}$ are precisely the ideals of $\mathcal{L}$.

References

[1] N. Avni, B. Klopsch, U. Onn, and C. Voll, *Representation zeta functions of compact p-adic analytic groups and arithmetic groups*, Duke Math. J. **162** (2013), no. 1, 111–197.

[2] A. Barvinok, *A course in convexity*, Graduate Studies in Mathematics, vol. 54, American Mathematical Society, Providence, RI, 2002.

[3] ———, *Integer points in polyhedra*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[4] E. Bierstone, D. Grigoriev, P. Milman, and J. Włodarczyk, *Effective Hironaka resolution and its complexity*, Asian J. Math. **15** (2011), no. 2, 193–228.

[5] E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302.

[6] B. Bories, *Igusa's p-adic local zeta function associated to a polynomial mapping and a polynomial integration measure*, Manuscripta Math. **138** (2012), no. 3-4, 395–417.

[7] N. Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989. Translated from the French, Reprint of the 1972 edition.

[8] C. J. Bushnell and I. Reiner, *Zeta functions of arithmetic orders and Solomon's conjectures*, Math. Z. **173** (1980), no. 2, 135–161.

[9] H. Cohen, *Number theory. Vol. I. Tools and Diophantine equations*, Graduate Texts in Mathematics, vol. 239, Springer, New York, 2007.

[10] D. A. Cox, J. Little, and D. O'Shea, *Using algebraic geometry*, Second, Graduate Texts in Mathematics, vol. 185, Springer, New York, 2005.

[11] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[12] W. A. de Graaf, *Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2*, J. Algebra **309** (2007), no. 2, 640–653.

[13] J. Denef, *On the degree of Igusa's local zeta function*, Amer. J. Math. **109** (1987), no. 6, 991–1008.

[14] ———, *Local zeta functions and Euler characteristics*, Duke Math. J. **63** (1991), no. 3, 713–721.

[15] ———, *Report on Igusa's local zeta function*, Astérisque **201-203** (1991), Exp. No. 741, 359–386 (1992). Séminaire Bourbaki, Vol. 1990/91.

[16] J. Denef and K. Hoornaert, *Newton polyhedra and Igusa's local zeta function*, J. Number Theory **89** (2001), no. 1, 31–64.

[17] J. Denef and F. Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc. **5** (1992), no. 4, 705–720.

[18] ———, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), no. 3, 505–537.

[19] M. P. F. du Sautoy, *The zeta function of* $\mathfrak{sl}_2(\mathbb{Z})$, Forum Math. **12** (2000), no. 2, 197–221.

[20] M. P. F. du Sautoy and F. J. Grunewald, *Analytic properties of zeta functions and subgroup growth*, Ann. of Math. (2) **152** (2000), no. 3, 793–833.

[21] M. P. F. du Sautoy and F. Loeser, *Motivic zeta functions of infinite-dimensional Lie algebras*, Selecta Math. (N.S.) **10** (2004), no. 2, 253–303.

[22] M. P. F. du Sautoy and A. Lubotzky, *Functional equations and uniformity for local zeta functions of nilpotent groups*, Amer. J. Math. **118** (1996), no. 1, 39–90.

[23] M. P. F. du Sautoy and G. Taylor, *The zeta function of* $\mathfrak{sl}_2$ *and resolution of singularities*, Math. Proc. Cambridge Philos. Soc. **132** (2002), no. 1, 57–73.

[24] M. P. F. du Sautoy and L. Woodward, *Zeta functions of groups and rings*, Lecture Notes in Mathematics, vol. 1925, Springer-Verlag, Berlin, 2008.

[25] K. Fukuda, *From the zonotope construction to the Minkowski addition of convex polytopes*, J. Symbolic Comput. **38** (2004), no. 4, 1261–1272.

[26] I. P. Goulden and D. M. Jackson, *Combinatorial enumeration*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1983.

[27] F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), no. 1, 185–223.

[28] J. Howald, M. Mustaţă, and C. Yuen, *On Igusa zeta functions of monomial ideals*, Proc. Amer. Math. Soc. **135** (2007), no. 11, 3425–3433 (electronic).

[29] J. ichi Igusa, *An introduction to the theory of local zeta functions*, AMS/IP Studies in Advanced Mathematics, vol. 14, Providence, RI: American Mathematical Society, 2000.

[30] I. Ilani, *Zeta functions related to the group* $\mathrm{SL}_2(\mathbb{Z}_p)$, Israel J. Math. **109** (1999), 157–172.

[31] N. Jacobson, *Lie algebras*, Dover Publications, Inc., New York, 1979. Republication of the 1962 original.

[32] N. M. Katz, *Review of ℓ-adic cohomology*, Motives (Seattle, WA, 1991), 1994, pp. 21–30.

[33] A. G. Khovanskii, *Newton polyhedra, and toroidal varieties*, Funkcional. Anal. i Priložen. **11** (1977), no. 4, 56–64, 96.

[34] ———, *Newton polyhedra, and the genus of complete intersections*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 1, 51–61.

[35] B. Klopsch and C. Voll, *Zeta functions of three-dimensional p-adic Lie algebras*, Math. Z. **263** (2009), no. 1, 195–210.

[36] H. Kraft and V. L. Popov, *Semisimple group actions on the three-dimensional affine space are linear*, Comment. Math. Helv. **60** (1985), no. 3, 466–479.

[37] A. G. Kushnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), no. 1, 1–31.

[38] A. Laeremans and W. Veys, *On the poles of maximal order of the topological zeta function*, Bull. London Math. Soc. **31** (1999), no. 4, 441–449.

[39] A. Lemahieu, D. Segers, and W. Veys, *On the poles of topological zeta functions*, Proc. Amer. Math. Soc. **134** (2006), no. 12, 3429–3436 (electronic).

[40] J. A. D. Loera, R. Hemmecke, J. Tauzer, and R. Yoshida, *Effective lattice point counting in rational convex polytopes*, J. Symbolic Comput. **38** (2004), no. 4, 1273–1302.

[41] F. Loeser, *Seattle lectures on motivic integration*, Algebraic geometry—Seattle 2005. Part 2, 2009, pp. 745–784.

[42] J. Nicaise, *An introduction to p-adic and motivic zeta functions and the monodromy conjecture*, Algebraic and analytic aspects of zeta functions and L-functions, 2010, pp. 141–166.

[43] M. Oka, *Principal zeta-function of nondegenerate complete intersection singularity*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **37** (1990), no. 1, 11–32.

[44] T. Rossmann, *Computing topological zeta functions of groups, algebras, and modules, II (preprint)* (2014). arXiv:1409.5044.

[45] D. Segal, *On the growth of ideals and submodules*, J. London Math. Soc. (2) **56** (1997), no. 2, 245–263.

[46] J.-P. Serre, *Lectures on $N_X(p)$*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 11, CRC Press, Boca Raton, FL, 2012.

[47] L. Solomon, *Zeta functions and integral representation theory*, Advances in Math. **26** (1977), no. 3, 306–326.

[48] B. Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996.

[49] A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. **37** (1976), no. 3, 253–262.

[50] W. Veys, *Determination of the poles of the topological zeta function for curves*, Manuscripta Math. **87** (1995), no. 4, 435–448.

[51] W. Veys and W. A. Zúñiga-Galindo, *Zeta functions for analytic mappings, log-principalization of ideals, and Newton polyhedra*, Trans. Amer. Math. Soc. **360** (2008), no. 4, 2205–2227.

[52] O. Villamayor, *Constructiveness of Hironaka's resolution*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 1–32.

[53] C. Voll, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), no. 2, 1181–1218.

[54] J. Włodarczyk, *Simple Hironaka resolution in characteristic zero*, J. Amer. Math. Soc. **18** (2005), no. 4, 779–822 (electronic).

[55] L. Woodward, *Zeta functions of groups: computer calculations and functional equations*, Ph.D. Thesis, 2005.

[56] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.

[57] W. A. Zúñiga-Galindo, *Local zeta functions supported on analytic submanifolds and Newton polyhedra*, Int. Math. Res. Not. IMRN **15** (2009), 2855–2898.