

Computing local zeta functions of groups, algebras, and modules

Tobias Rossmann

Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany

Current address: Department of Mathematics, University of Auckland, Auckland, New Zealand

We develop a practical method for computing local zeta functions of groups, algebras, and modules in fortunate cases. Using our method, we obtain a complete classification of generic local representation zeta functions associated with unipotent algebraic groups of dimension at most six. We also determine the generic local subalgebra zeta functions associated with $\mathfrak{gl}_2(\mathbf{Q})$. Finally, we introduce and compute examples of graded subobject zeta functions.

1. Introduction

Zeta functions counting subobjects and representations. By considering associated Dirichlet series, various algebraic counting problems give rise to a *global zeta function* $Z(s)$ which admits a natural Euler product factorisation $Z(s) = \prod_p Z_p(s)$ into *local zeta functions* $Z_p(s)$ indexed by rational primes p . For example, $Z(s)$ could be the Dirichlet series enumerating subgroups of finite index within a finitely generated nilpotent group and $Z_p(s)$ might enumerate those subgroups of p -power index only (see [26]); in the special case of the infinite cyclic group, we then recover the classical Euler factorisation $\zeta(s) = \prod_p 1/(1 - p^{-s})$ of the Riemann zeta function.

This article is concerned with three types of counting problems and associated zeta functions; all of these problems arose from (and remain closely related to) enumerative problems for nilpotent groups.

- Enumerate subalgebras of finite additive index of a possibly non-associative algebra, e.g. a Lie algebra (possibly taking into account an additive grading). (See [26].)
- Enumerate submodules of finite additive index under the action of an integral matrix algebra. (See [43].)
- Enumerate twist-isoclasses of finite-dimensional complex representations of a finitely generated nilpotent group. (See [27, 46].)

2010 Mathematics Subject Classification. 11M41, 20F69, 20G30, 20F18, 20C15.

Keywords. Subgroup growth, representation growth, zeta functions, unipotent groups, Lie algebras.

This work was supported by the DFG Priority Programme “Algorithmic and Experimental Methods in Algebra, Geometry and Number Theory” (SPP 1489).

Generic local zeta functions. Each of the preceding three counting problems provides us with a global zeta function $Z(s)$ (namely the associated Dirichlet series) and a factorisation $Z(s) = \prod_p Z_p(s)$ as above. The goal of this article is to compute the *generic local zeta functions* $Z_p(s)$ at least in favourable situations—that is, we seek to simultaneously determine $Z_p(s)$ for almost all p using a single finite computation. To see why this is a sensible problem, we first recall some theory.

In the cases of interest to us, each $Z_p(s)$ will be a rational function in p^{-s} over \mathbf{Q} . In particular, the task of computing one local zeta function $Z_p(s)$ using exact arithmetic is well-defined. Regarding the behaviour of $Z_p(s)$ under variation of p , in all three cases from above, sophisticated results from p -adic integration imply the existence of schemes V_1, \dots, V_r and rational functions $W_1, \dots, W_r \in \mathbf{Q}(X, Y)$ such that for almost all primes p ,

$$Z_p(s) = \sum_{i=1}^r \#V_i(\mathbf{F}_p) \cdot W_i(p, p^{-s}); \quad (1.1)$$

for more details, see Theorem 4.1 below. While constructive proofs of (1.1) are known, they are generally impractical due to their reliance on resolution of singularities.

Previous work: computing topological zeta functions. In [35–37], the author developed practical methods for computing so-called topological zeta functions associated with the above counting problems; these zeta functions are derived from generic local ones by means of a termwise limit “ $p \rightarrow 1$ ” applied to a formula (1.1). Due to their reliance on non-degeneracy conditions for associated families of polynomials, the author’s methods for computing topological zeta functions do not apply in all cases. However, whenever they are applicable, as we will explain below, they come close to producing an *explicit* formula (1.1).

Computing generic local zeta functions. In general, we understand the task of computing $Z_p(s)$ for almost all p to be the explicit construction of V_i and W_i as in (1.1). While this seems to be the only adequate general notion of “computing” generic local zeta functions, we will often be more ambitious in practice.

Uniformity Problem. *Decide if there exists $W \in \mathbf{Q}(X, Y)$ such that $Z_p(s) = W(p, p^{-s})$ for almost all primes p ; in that case, we call $(Z_p(s))_{p \text{ prime}}$ **uniform**. Find W if it exists.*

The term “uniformity” is taken from [18, §1.2.4]. In practice, a weaker, non-constructive form of the Uniformity Problem which merely asks for the existence of W as above is often easier to solve. For example, if $Z_p(s)$ is the zeta function enumerating subgroups (or normal subgroups) of finite index in the free nilpotent pro- p group of some fixed finite rank (independent of p) and class 2, then $(Z_p(s))_{p \text{ prime}}$ is shown to be uniform in [26, Thm 2] even though no explicit construction of a rational function W is given.

For many cases of interest, a rational function W as in the Uniformity Problem exists, see e.g. most examples in [18]. However, no conceptual explanation as to why this is so seems to be known beyond explicit computations.

Woodward [50] used computer-assisted calculations to solve the Uniformity Problem for a large number of subalgebra and ideal zeta functions of nilpotent Lie algebras. Unfortunately, few details on his computations are available, rendering them rather difficult to reproduce.

Results. While explicit formulae (1.1) have been obtained for specific examples and even certain infinite families of these, all known general constructions of V_i and W_i as in (1.1) are impractical. In full generality, we thus regard the Uniformity Problem as too ambitious a task. In the present article, we extend the author’s work on explicit, combinatorially defined formulae (1.1) (see [35–37]) in order to provide practical solutions to the Uniformity Problem in fortunate cases. We will also consider computations of generic local zeta functions in cases where no W as above exists.

As the following list illustrates, the method developed here can be used to compute a substantial number of interesting new examples of generic local zeta functions:

- We completely determine the generic local representation zeta functions associated with unipotent algebraic groups of dimension at most six (§8, Table 1).
- We compute the generic local subalgebra zeta functions associated with $\mathfrak{gl}_2(\mathbf{Q})$; this constitutes only the second instance (after $\mathfrak{sl}_2(\mathbf{Q})$) where such zeta functions associated with an insoluble Lie algebra have been computed (§9.1, Theorem 9.1).
- We compute the generic local submodule zeta functions for the natural action of the group of upper unitriangular integral $n \times n$ -matrices (or, equivalently, the nilpotent associative algebra of strictly upper triangular integral $n \times n$ -matrices) for $n \leq 5$ (§9.4, Theorem 9.5).
- We compute the graded subalgebra and ideal zeta functions associated with specific \mathbf{Q} -forms of each of the 26 “fundamental graded” Lie algebras of dimension at most six over \mathbf{C} (§10, Tables 2–3).

Outline. In §2, we recall definitions of the subobject and representation zeta functions of concern to us. In §3, as a variation of established subalgebra and ideal zeta functions, we discuss graded versions of these zeta functions. In §4, we consider formulae such as (1.1) both in theory and as provided by the author’s previous work. Our work on the Uniformity Problem then proceeds in two steps. First, in §5, we consider the symbolic determination of numbers such as the $\#V_i(\mathbf{F}_p)$ in (1.1) as a function of p . Thereafter, in §6, we discuss the explicit computation of the rational functions W_i as provided by [35–37]; a key role will be played by algorithms of Barvinok et al. [5–7] surrounding generating functions of rational polyhedra. In §7, we consider “reduced representation zeta functions” in the spirit of Evseev’s work [20]; while these functions turn out to be trivial, they provide us with a simple necessary condition for the correctness of calculations. Finally, examples of generic local zeta functions are the subject of §§8–10.

Acknowledgements

I would like to thank Christopher Voll for interesting discussions and the anonymous referee for helpful comments.

Notation

The symbol “ \subset ” indicates not necessarily proper inclusion. For the remainder of this article, let k be a number field with ring of integers \mathfrak{o} . We write \mathcal{V}_k for the set of non-Archimedean places of k . For $v \in \mathcal{V}_k$, we denote by k_v the v -adic completion of k and by \mathfrak{o}_v the valuation ring of k_v . We further let $\mathfrak{p}_v \in \text{Spec}(\mathfrak{o})$ denote the prime ideal corresponding to $v \in \mathcal{V}_k$ and write $q_v = |\mathfrak{o}/\mathfrak{p}_v|$. Finally, we let $|\cdot|_v$ denote the absolute value on k_v with $|\pi|_v = q_v^{-1}$ for $\pi \in \mathfrak{p}_v \setminus \mathfrak{p}_v^2$.

We let \mathbf{Q}_p and \mathbf{Z}_p denote the field of p -adic numbers and ring of p -adic integers, respectively. By a p -adic field, we mean a finite extension of \mathbf{Q}_p . For a p -adic field K , let \mathfrak{D}_K denote the valuation ring of K and let \mathfrak{P}_K denote the maximal ideal of \mathfrak{D}_K . We write $q_K = |\mathfrak{D}_K/\mathfrak{P}_K|$.

2. Established zeta functions of groups, algebras, and modules

2.1. Subalgebra and ideal zeta functions

Following [26] (cf. [35, §2.1]), for a commutative ring R and a (possibly non-associative) R -algebra A , we formally define the **subalgebra zeta function** of A to be

$$\zeta_A^{\leq}(s) = \sum_{U} |A : U|^{-s},$$

where U ranges over the R -subalgebras of A such that the R -module quotient A/U has finite cardinality $|A : U|$. Additional hypotheses (which are satisfied in our applications below) ensure that the number $a_n(A)$ of R -subalgebras of index n of A is finite for every $n \geq 1$ and, in addition, $a_n(A)$ grows at most polynomially as a function of n . Under these assumptions, $\zeta_A^{\leq}(s)$ defines an analytic function in some complex right half-plane.

Now let \mathcal{A} be a finite-dimensional possibly non-associative k -algebra, where k is a number field as above. Choose an \mathfrak{o} -form A of \mathcal{A} whose underlying \mathfrak{o} -module is free. For $v \in \mathcal{V}_k$, let $A_v := A \otimes_{\mathfrak{o}} \mathfrak{o}_v$, regarded as an \mathfrak{o}_v -algebra. We then have an Euler product $\zeta_A^{\leq}(s) = \prod_{v \in \mathcal{V}_k} \zeta_{A_v}^{\leq}(s)$; see [35, Lem. 2.3]. While the **global zeta function** $\zeta_A^{\leq}(s)$ is an analytic object, as we will recall below, the **local zeta functions** $\zeta_{A_v}^{\leq}(s)$ are algebro-geometric in nature. Note that up to discarding finitely many elements, the family $(\zeta_{A_v}^{\leq}(s))_{v \in \mathcal{V}_k}$ of local zeta functions only depends on \mathcal{A} and not on the \mathfrak{o} -form A .

If, instead of enumerating subalgebras, we consider ideals, we obtain the global and local **ideal zeta functions** $\zeta_A^{\triangleleft}(s)$ and $\zeta_{A_v}^{\triangleleft}(s)$ of A , respectively; these are also linked by an Euler product as above.

2.2. Submodule zeta functions

Submodule zeta functions were introduced by Solomon [43] in the context of semisimple associative algebras. In the following generality (based upon [35, §2.1]), they also generalise ideal zeta functions of algebras. For a commutative ring R , an R -module \mathbf{V} , and a set $\Omega \subset \text{End}_R(\mathbf{V})$, we formally define the **submodule zeta function** of Ω acting on \mathbf{V} to be

$$\zeta_{\Omega \curvearrowright \mathbf{V}}(s) = \sum_{\mathbf{U}} |\mathbf{V} : \mathbf{U}|^{-s},$$

where \mathbf{U} ranges over the Ω -invariant R -submodules of \mathbf{V} with finite R -module quotients \mathbf{V}/\mathbf{U} . The name “submodule zeta function” is justified by the observation that we are free to replace Ω by its enveloping unital associative algebra within $\text{End}_R(\mathbf{V})$.

Let V be a finite-dimensional vector space over k and let $\Omega \subset \text{End}_k(V)$ be given. Choose an \mathfrak{o} -form \mathbf{V} of V which is free as an \mathfrak{o} -module. Furthermore, choose a finite set $\Omega \subset \text{End}_{\mathfrak{o}}(\mathbf{V})$ which generates the same unital subalgebra of $\text{End}_k(V)$ as Ω . Writing $\mathbf{V}_v = \mathbf{V} \otimes_{\mathfrak{o}} \mathfrak{o}_v$, we obtain an Euler product $\zeta_{\Omega \curvearrowright \mathbf{V}}(s) = \prod_{v \in \mathcal{V}_k} \zeta_{\Omega \curvearrowright \mathbf{V}_v}(s)$; as in §2.1, up to discarding finitely many factors, the collection of local zeta functions on the right-hand side of this product only depends on (Ω, V) and not on the choice of (Ω, \mathbf{V}) .

2.3. Representation zeta functions associated with unipotent groups

Following [27, 44], for a topological group G , we let $\tilde{r}_n(G)$ denote the number of continuous irreducible representations $G \rightarrow \text{GL}_n(\mathbf{C})$ counted up to equivalence and tensoring with continuous 1-dimensional complex representations. We formally define the **(twist) representation zeta function** of G to be

$$\zeta_G^{\text{irr}}(s) = \sum_{n=1}^{\infty} \tilde{r}_n(G) n^{-s}.$$

Let \mathbf{G} be a unipotent algebraic group over k ; see [10, Ch. IV] for background. Let U_n denote the group scheme of upper unitriangular $n \times n$ -matrices. We choose an embedding of \mathbf{G} into some $U_n \otimes k$ and let $\mathbf{G} \leq U_n \otimes \mathfrak{o}$ be the associated \mathfrak{o} -form of \mathbf{G} (viz. the scheme-theoretic closure of \mathbf{G} within $U_n \otimes \mathfrak{o}$). By [44, Prop. 2.2], the Euler product $\zeta_{\mathbf{G}(\mathfrak{o})}^{\text{irr}}(s) = \prod_{v \in \mathcal{V}_k} \zeta_{\mathbf{G}(\mathfrak{o}_v)}^{\text{irr}}(s)$ connects the representation zeta function of the discrete group $\mathbf{G}(\mathfrak{o})$ and those of the pro- p_v groups $\mathbf{G}(\mathfrak{o}_v)$, where p_v is the rational prime contained in \mathfrak{p}_v .

2.4. Motivation: zeta functions of nilpotent groups

We briefly recall the original motivation for the study of subalgebra and ideal zeta function from [26] and representation zeta functions in [27, 46] (cf. [44]). For any topological group G , the **subgroup zeta function** $\zeta_G^{\leq}(s)$ (resp. the **normal subgroup zeta function** $\zeta_G^{\triangleleft}(s)$) of G is formally defined to be $\sum_H |G : H|^{-s}$, where H ranges of the closed subgroups (resp. closed normal subgroups) of G of finite index. Let G be a discrete torsion-free finitely generated nilpotent group. Then $\zeta_G^{\leq}(s) = \prod_p \zeta_{\hat{G}_p}^{\leq}(s)$, where p ranges over primes and \hat{G}_p denotes the pro- p completion of G . Moreover, the global

and local zeta functions $\zeta_G^{\leq}(s)$ and $\zeta_{\hat{G}_p}^{\leq}(s)$ all converge in some complex right half-plane. Analogous statements hold for the normal subgroup and representation zeta functions of G .

Apart from finitely many exceptions, the local subobject and representation zeta functions attached to G are special cases of those in §§2.1–2.2. Recall that the Mal'cev correspondence attaches a finite-dimensional nilpotent Lie \mathbf{Q} -algebra, \mathcal{L} say, to G . As explained in [26], if L is a \mathbf{Z} -form of \mathcal{L} which is finitely generated as a \mathbf{Z} -module, then $\zeta_{\hat{G}_p}^{\leq}(s) = \zeta_{L \otimes \mathbf{Z}_p}^{\leq}(s)$ and $\zeta_{\hat{G}_p}^{\triangleleft}(s) = \zeta_{L \otimes \mathbf{Z}_p}^{\triangleleft}(s)$ for almost all p . Moreover, if \mathbf{G} is the unipotent algebraic group over \mathbf{Q} with Lie algebra \mathcal{L} and if G is a \mathbf{Z} -form of \mathbf{G} arising from an embedding $\mathbf{G} \leq \mathbf{U}_n \otimes \mathbf{Q}$, then $\hat{G}_p = G(\mathbf{Z}_p)$ for almost all primes p (see [44]).

3. Graded subalgebra and ideal zeta functions

In this section, we introduce variations of the subalgebra and ideal zeta functions from §2.1 which take into account a given additive grading of the algebra under consideration.

3.1. Definitions

Let R be a commutative ring and let A be a possibly non-associative R -algebra. Further suppose that we are given a direct sum decomposition

$$A = A_1 \oplus \cdots \oplus A_r \tag{3.1}$$

of R -modules. As usual, an R -submodule $U \leq A$ is **homogeneous** if it decomposes as $U = U_1 \oplus \cdots \oplus U_r$ for R -submodules $U_i \leq A_i$ for $i = 1, \dots, r$. We formally define the **graded subalgebra zeta function** of A with respect to the decomposition (3.1) to be

$$\zeta_A^{\text{gr}\leq}(s) = \sum_U |A : U|^{-s},$$

where U ranges over the homogeneous R -subalgebras of A such that the R -module quotient A/U is finite. We also define the **graded ideal zeta function** $\zeta_A^{\text{gr}\triangleleft}(s)$ in the evident way. Note that we do not require (3.1) to be compatible with the given multiplication in A . As in the non-graded context, given a finite-dimensional possibly non-associative k -algebra \mathcal{A} together with a vector space decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r$, we obtain associated global and local graded subalgebra and ideal zeta functions generalising those from §2.1 by choosing appropriate \mathfrak{o} -forms.

Example 3.1. Let $A = \mathbf{Z}^{n_1} \oplus \cdots \oplus \mathbf{Z}^{n_r}$ be regarded as an abelian Lie \mathbf{Z} -algebra for $n_1, \dots, n_r \geq 1$. It follows from the well-known non-graded case ($r = 1$; see [26, Prop. 1.1]) that $\zeta_A^{\text{gr}\leq}(s) = \prod_{i=1}^r \prod_{j=0}^{n_i-1} \zeta(s-j)$, where ζ denotes the Riemann zeta function.

Remark 3.2. Let R, V , and $\Omega \subset \text{End}_R(V)$ be as in §2.2. Fix an R -module decomposition $V = V_1 \oplus \cdots \oplus V_r$. In analogy to the above, we define the **graded submodule zeta function** $\zeta_{\Omega \curvearrowright V}^{\text{gr}}$ of Ω by enumerating homogeneous Ω -invariant R -submodules of V .

3.2. Reminder: graded Lie algebras

Let R be a commutative Noetherian ring. All Lie R -algebras in the following are assumed to be finitely generated as R -modules. Recall that an **(N-)graded Lie algebra** over R is a Lie R -algebra \mathfrak{g} together with a decomposition $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ into R -submodules $\mathfrak{g}_i \leq \mathfrak{g}$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \leq \mathfrak{g}_{i+j}$ for all $i, j \geq 1$. Since \mathfrak{g} is Noetherian as an R -module, $\mathfrak{g}_i = 0$ for sufficiently large i whence such an algebra \mathfrak{g} is nilpotent. Following [31, §2, Def. 1], we say that \mathfrak{g} is **fundamental** if $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for all $i \geq 1$. If $R = \mathbf{R}$ or $R = \mathbf{C}$, then the fundamental graded Lie R -algebras of dimension at most 7 have been classified in [31]. In the case of dimension at most 5, the classification in [31] is in fact valid over any field of characteristic zero; see [31, §2.2, Rem. 1].

Let \mathfrak{g} be a finite-dimensional Lie algebra over a field. Let $\mathfrak{g} = \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \dots$ be the lower central series of \mathfrak{g} . As is well-known, commutation in \mathfrak{g} endows $\text{gr}(\mathfrak{g}) := \bigoplus_{i=1}^{\infty} \mathfrak{g}^i / \mathfrak{g}^{i+1}$ with the structure of a graded Lie algebra; note that $\text{gr}(\mathfrak{g})$ is fundamental by construction. We call $\text{gr}(\mathfrak{g})$ the **graded Lie algebra associated with \mathfrak{g}** .

The study of graded zeta functions seems quite natural in the context of nilpotent Lie algebras. It would be interesting to find group-theoretic interpretations, in the spirit of §2.4, of such zeta functions associated with graded nilpotent Lie algebras.

3.3. Graded subobject zeta functions as p -adic integrals

In order to carry out explicit computations of local graded subobject zeta functions, we will use the following straightforward variation of [15, §5]; we only spell out the enumeration of graded subalgebras, the case of ideals being analogous.

Theorem 3.3. *Let \mathfrak{D} be the valuation ring of a non-Archimedean local field. Let A be a (possibly non-associative) \mathfrak{D} -algebra whose underlying \mathfrak{D} -module is free with basis $\mathbf{a} = (a_1, \dots, a_d)$. Let $0 = \beta_1 < \dots < \beta_{r+1} = d$ and decompose $A = A_1 \oplus \dots \oplus A_r$ by setting $A_i = \mathfrak{D}a_{1+\beta_i} \oplus \dots \oplus \mathfrak{D}a_{\beta_{i+1}}$.*

Let T denote the \mathfrak{D} -module of block diagonal upper triangular $d \times d$ -matrices over \mathfrak{D} with block sizes $\beta_2 - \beta_1, \dots, \beta_{r+1} - \beta_r$. Let $M(\mathbf{X})$ be the generic matrix of the same shape over \mathfrak{D} ; in other words,

$$M(\mathbf{X}) = \text{diag} \left(\begin{bmatrix} X_{1,1} & \dots & X_{1,\beta_2} \\ & \ddots & \vdots \\ & & X_{\beta_2,\beta_2} \end{bmatrix}, \dots, \begin{bmatrix} X_{1+\beta_r,1+\beta_r} & \dots & X_{1+\beta_r,d} \\ & \ddots & \vdots \\ & & X_{d,d} \end{bmatrix} \right).$$

Let $R = \mathfrak{D}[\mathbf{X}]$ and let $\star: R^d \times R^d \rightarrow R^d$ be induced via base extension by multiplication in A with respect to \mathbf{a} . Let $F \subset R$ consist of all entries of all d -tuples $(M_i(\mathbf{X}) \star M_j(\mathbf{X})) \text{adj}(M(\mathbf{X}))$ for $1 \leq i, j \leq d$, where $\text{adj}(M(\mathbf{X}))$ denotes the adjugate matrix of $M(\mathbf{X})$ and $M_i(\mathbf{X})$ the i th row of $M(\mathbf{X})$. Define $V = \{\mathbf{x} \in T : \det(M(\mathbf{x})) \mid f(\mathbf{x}) \text{ for all } f \in F\}$. Let q denote the residue field size of \mathfrak{D} , let μ denote the normalised Haar measure on $T \approx \mathfrak{D}^{\sum_{i=1}^r \binom{\beta_{i+1} - \beta_i + 1}{2}}$, and let $|\cdot|$ denote the absolute value on K such

that $|\pi| = q^{-1}$ for any uniformiser π . Then

$$\zeta_{\mathbf{A}}^{\text{gr}\leq}(s) = (1 - q^{-1})^{-d} \int_V \prod_{i=1}^r \prod_{j=1}^{\beta_{i+1} - \beta_i} |x_{j+\beta_i, j+\beta_i}|^{s-j} d\mu(\mathbf{x}). \quad \blacklozenge \quad (3.2)$$

Remark 3.4. As in [15, §5], a matrix $\mathbf{x} \in T$ belongs to the set V in Theorem 3.3 if and only if its row span is a subalgebra of \mathfrak{D}^d , regarded as an algebra via the given identification $\mathbf{A} = \mathfrak{D}^d$.

The following illustrates Theorem 3.3 for an infinite family of graded algebras.

Proposition 3.5. Let $n \geq 1$ and let $\mathfrak{m}(n) = \mathfrak{m}_1(n) \oplus \cdots \oplus \mathfrak{m}_n(n)$ be the graded Lie \mathbf{Z} -algebra of additive rank $n+1$ and nilpotency class n with $\mathfrak{m}_1(n) = \mathbf{Z}e_0 \oplus \mathbf{Z}e_1$, $\mathfrak{m}_i(n) = \mathbf{Z}e_i$ for $i = 2, \dots, n$, and non-trivial commutators $[e_0, e_i] = e_{i+1}$ for $1 \leq i \leq n-1$. Let k be a number field with ring of integers \mathfrak{o} . Then for each $v \in \mathcal{V}_k$,

$$\zeta_{\mathfrak{m}(n) \otimes \mathfrak{o}_v}^{\text{gr}\triangleleft}(s) = 1 / ((1 - q_v^{-s})(1 - q_v^{1-s})(1 - q_v^{-3s})(1 - q_v^{-4s}) \cdots (1 - q_v^{-(n+1)s})),$$

where $\mathfrak{m}(n) \otimes \mathfrak{o}_v$ is regarded as an \mathfrak{o}_v -algebra. Denoting the Dedekind zeta function of k by $\zeta_k(s)$, we thus have

$$\zeta_{\mathfrak{m}(n) \otimes \mathfrak{o}}^{\text{gr}\triangleleft}(s) = \zeta_k(s)\zeta_k(s-1)\zeta_k(3s)\zeta_k(4s) \cdots \zeta_k((n+1)s).$$

Proof. It is an elementary consequence of Theorem 3.3 and Remark 3.4 (both applied to the enumeration of ideals instead of subalgebras) that for any $v \in \mathcal{V}_k$,

$$\begin{aligned} \zeta_{\mathfrak{m}(n) \otimes \mathfrak{o}_v}^{\text{gr}\triangleleft}(s) &= (1 - q_v^{-1})^{-n-1} \\ &\times \int_V |x_1|_v^{s-1} |x_3|_v^{s-2} |y_1|_v^{s-1} \cdots |y_{n-1}|_v^{s-1} d\mu(x_1, x_2, x_3, y_1, \dots, y_{n-1}), \end{aligned}$$

where $V = \{(x_1, x_2, x_3, y_1, \dots, y_{n-1}) \in \mathfrak{o}_v^{n+2} : y_{n-1} \mid y_n \mid \cdots \mid y_1 \mid x_1, x_2, x_3\}$; indeed, $(x_1, \dots, y_{n-1}) \in V$ if and only if the row span of $\text{diag}(\begin{bmatrix} x_1 & x_2 \\ x_3 & \end{bmatrix}, y_1, \dots, y_{n-1})$ is an ideal of $\mathfrak{m}(n) \otimes \mathfrak{o}_v$ (identified with \mathfrak{o}_v^{n+1} via (e_0, \dots, e_n)). Define a bianalytic bijection

$$\begin{aligned} \varphi: (k_v^\times)^{n+2} \rightarrow (k_v^\times)^{n+2}, \quad (x_1, x_2, x_3, y_1, \dots, y_{n-1}) \mapsto & (x_1 y_1 \cdots y_{n-1}, x_2 y_1 \cdots y_{n-1}, \\ & x_3 y_1 \cdots y_{n-1}, \\ & y_1 \cdots y_{n-1}, \\ & y_2 \cdots y_{n-1}, \dots, y_{n-1}); \end{aligned}$$

note that the Jacobian determinant of φ is $\det \varphi'(x_1, x_2, x_3, y_1, \dots, y_{n-1}) = y_1^3 y_2^4 \cdots y_{n-1}^{n+1}$. Since $V \cap (k_v^\times)^{n+2} = \varphi(\mathfrak{o}_v^{n+2} \cap (k_v^\times)^{n+2})$ and $\mu(k_v^n \setminus (k_v^\times)^n) = 0$, by performing a change of variables using φ and using the well-known fact $\int_{\mathfrak{o}_v} |z|_v^s d\mu(z) = (1 - q_v^{-1}) / (1 - q_v^{-1-s})$,

$$\begin{aligned} \zeta_{\mathfrak{m}(n) \otimes \mathfrak{o}_v}^{\text{gr}\triangleleft}(s) &= (1 - q_v^{-1})^{-n-1} \int_{\mathfrak{o}_v^{n+2}} |x_1|_v^{s-1} |x_3|_v^{s-2} |y_1|_v^{3s-1} \cdots |y_{n-1}|_v^{(n+1)s-1} d\mu(x_1, \dots, y_{n-1}) \\ &= 1 / ((1 - q_v^{-s})(1 - q_v^{1-s})(1 - q_v^{-3s})(1 - q_v^{-4s}) \cdots (1 - q_v^{-(n+1)s})). \end{aligned}$$

The final claim follows by taking the product over all $v \in \mathcal{V}_k$. \blacklozenge

Remark. To the author’s knowledge, not a single example of a non-graded subobject zeta function of a nilpotent Lie algebra of nilpotency class ≥ 5 is known explicitly.

Integrals such as those in (3.2) are special cases of those associated with “toric data” in [36, §3]. Hence, the author’s methods for manipulating such integrals as developed in [36] apply directly without modification, as do the techniques explained below.

4. Explicit formulae

4.1. Theory: local zeta functions of Denef type

The following is a variation of the terminology employed in [35, §5.2]. As before, we assume that k is a fixed number field. Suppose that we are given a collection $Z = (Z_K(s))_K$ of analytic functions of a complex variable s (each defined in some right half-plane) indexed by p -adic fields $K \supset k$ (up to k -isomorphism). We say that Z is of **Denef type** if there exist a finite set $S \subset \mathcal{V}_k$, k -varieties V_1, \dots, V_r , and rational functions $W_1, \dots, W_r \in \mathbf{Q}(X, Y)$ such that for all $v \in \mathcal{V}_k \setminus S$ and all finite extensions K/k_v ,

$$Z_K(s) = \sum_{i=1}^r \#\bar{V}_i(\mathfrak{D}_K/\mathfrak{P}_K) \cdot W_i(q_K, q_K^{-s}) \quad (4.1)$$

is an identity of analytic functions; here, we wrote $\bar{V}_i = V_i \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{p}_v$ for a fixed but arbitrary \mathfrak{o} -model V_i of V_i .

The following result formalises our discussion surrounding (1.1) from the introduction; it summarises [15, §§2–3] (cf. [35, Thm 5.16]) and [44, Thm A].

Theorem 4.1. *Let $(Z_K(s))_K$ be one of the following collections of local zeta functions indexed by p -adic fields $K \supset k$ (up to k -isomorphism).*

- (i) $Z_K(s) = \zeta_{\mathbf{A} \otimes_{\mathfrak{o}} \mathfrak{D}_K}^{\leq}(s)$ or $Z_K(s) = \zeta_{\mathbf{A} \otimes_{\mathfrak{o}} \mathfrak{D}_K}^{\triangleleft}(s)$ (resp. $Z_K(s) = \zeta_{\mathbf{A} \otimes_{\mathfrak{o}} \mathfrak{D}_K}^{\text{gr} \leq}(s)$ or $Z_K(s) = \zeta_{\mathbf{A} \otimes_{\mathfrak{o}} \mathfrak{D}_K}^{\text{gr} \triangleleft}(s)$), where \mathbf{A} is an \mathfrak{o} -form of a finite-dimensional (possibly non-associative) k -algebra as in §2.1 or §3.1, respectively.
- (ii) $Z_K(s) = \zeta_{\Omega \curvearrowright (\mathbf{V} \otimes_{\mathfrak{o}} \mathfrak{D}_K)}(s)$, where Ω and \mathbf{V} are as in §2.2.
- (iii) $Z_K(s) = \zeta_{\tilde{\mathbf{G}}(\mathfrak{D}_K)}^{\text{irr}}(s)$, where \mathbf{G} is an \mathfrak{o} -form of a unipotent algebraic group over k as in §2.3.

Then $(Z_K(s))_K$ is of Denef type.

The known proofs of Theorem 4.1 are constructive but impractical due to their reliance on resolution of singularities. We note that the exclusion of finitely many primes implicit in Theorem 4.1 is one of the main reasons for our focus on *generic* local zeta functions.

4.2. By-products of the computation of topological zeta functions

The computation of topological zeta functions is often considerably easier than that of local ones. In [35–37], the author developed practical methods for computing topological zeta functions associated with the local zeta functions in Theorem 4.1; these methods are not algorithms because they may fail if certain non-degeneracy conditions are violated. While we will recall the specific notion of non-degeneracy in question in §4.3, we will essentially treat the methods from [35–37] as “black boxes”. In particular, from now on, we will assume the validity of the following.

Assumption 4.2. *In the setting of Theorem 4.1, the method from [36, §4] (resp. [37, §5.4]) for computing topological subalgebra and submodule zeta functions (resp. topological representation zeta functions) succeeds.*

Remark 4.3. The author is unaware of a useful intrinsic characterisation of those groups, algebras, and modules such that Assumption 4.2 is satisfied. The local zeta functions in Theorem 4.1 can be described in terms of p -adic integrals associated with a collection of polynomials; see, in particular, Theorem 3.3. A sufficient condition for the validity of Assumption 4.2 is (global) non-degeneracy of said collection of polynomials in the sense of [35, §4.2] (cf. [36, Lem. 5.7] and [37, §5.4.1]); see §4.3 below for a definition.

The first stages of the methods for computing topological zeta functions associated with the local zeta functions in Theorem 4.1, as described in [36, 37], come close to constructing an explicit formula (4.1). In detail, using [35, Thm 4.10] (see [36, Thm 5.8] and [37, Thm 5.9]), whenever they succeed, these methods derive a formula (4.1) such that the following Assumptions 4.4–4.5 are satisfied.

We write $\mathbf{T}^n := \text{Spec}(\mathbf{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$. For a commutative ring R , we write $\mathbf{T}_R^n := \mathbf{T}^n \otimes R$ and we identify $\mathbf{T}^n(R) = (R^\times)^n$. In particular, \mathbf{T}_k^n is the split algebraic torus of dimension n over k .

Assumption 4.4. *Each V_i in (4.1) is given as an explicit subvariety of some $\mathbf{T}_k^{n_i}$ defined by the vanishing of a finite number of Laurent polynomials and the non-vanishing of a single Laurent polynomial.*

The appearance of tori is closely related to the previously mentioned concept of non-degeneracy; some further details will again be given in §4.3.

Assumption 4.5. *Up to multiplication by explicitly given rational functions of the form $(X - 1)^a X^b$ (for suitable $a, b \in \mathbf{Z}$), each W_i in (4.1) is described explicitly in terms of generating functions associated with half-open cones and convex polytopes.*

We will clarify the deliberately vague formulation of Assumption 4.5 in §6.

In summary, whenever they apply, the methods for computing topological zeta functions in [36, 37] fall short of “constructing” an explicit formula (4.1) only in the sense that the W_i are characterised combinatorially instead of being explicitly given, say as fractions of polynomials.

In §§5–6, assuming the validity of Assumptions 4.2, 4.4, and 4.5, we will develop techniques for performing further computations with a formula of the form (4.1) with a view towards solving the Uniformity Problem from the introduction in fortunate cases.

4.3. Reminder: non-degeneracy

In order to shed some light on Assumptions 4.2 (via Remark 4.3) and 4.4, we now briefly recall the notion of non-degeneracy from [35, §4.2].

Initial forms. Let $f \in k[\mathbf{X}^{\pm 1}] := k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, say $f = \sum_{\alpha \in \mathbf{Z}^n} c_\alpha \mathbf{X}^\alpha$, where $c_\alpha \in k$, $c_\alpha = 0$ for almost all $\alpha \in \mathbf{Z}^n$, and $\mathbf{X}^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. The **Newton polytope** of f is the convex hull of $\{\alpha \in \mathbf{Z}^n : c_\alpha \neq 0\}$ within \mathbf{R}^n . Suppose that $f \neq 0$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbf{R}^n . Given $\omega \in \mathbf{R}^n$, let $m(f, \omega) \in \mathbf{R}$ be the minimal value attained by $\langle \alpha, \omega \rangle$ for $\alpha \in \mathbf{Z}^n$ with $c_\alpha \neq 0$. Define the **initial form** $\text{in}_\omega(f) \in k[\mathbf{X}^{\pm 1}]$ of f in the direction ω to be the sum of those $c_\alpha \mathbf{X}^\alpha$ ($\alpha \in \mathbf{Z}^n$) with $c_\alpha \neq 0$ and $\langle \alpha, \omega \rangle = m(f, \omega)$.

Non-degeneracy. Fix an algebraic closure \bar{k} of k . We say that a collection (f_1, \dots, f_r) of non-zero elements of $k[\mathbf{X}^{\pm 1}]$ is **(globally) non-degenerate** [35, Def. 4.2] if the following holds: for all $\omega \in \mathbf{R}^n$ and all $J \subset \{1, \dots, r\}$, if $\mathbf{x} \in \mathbf{T}^n(\bar{k})$ satisfies $\text{in}_\omega(f_j)(\mathbf{x}) = 0$ for all $j \in J$, then the rank of

$$\left[\frac{\partial \text{in}_\omega(f_j)(\mathbf{x})}{\partial X_i} \right]_{\substack{i=1, \dots, n, \\ j \in J}}$$

is $|J|$.

Remark.

- (i) The author’s computations in [35–37] actually rely on a slightly less restrictive relative notion of non-degeneracy; see [35, Def. 4.2(i)]. For a discussion of related notions of non-degeneracy in the literature, see [35, §§4.2, 4.4].
- (ii) The different tuples of the form $(\text{in}_\omega(f_1), \dots, \text{in}_\omega(f_r))$ for $\omega \in \mathbf{R}^n$ correspond naturally to the faces of the Newton polytope of $f_1 \cdots f_r$; see [35, §4.1].

Subvarieties of tori. We may now indicate the origin of Assumption 4.4. Suppose that the $Z_K(s)$ in (4.1) are all given by a “suitable” (see [35, Def. 4.6]) type of p -adic integral defined in terms of a fixed family (f_1, \dots, f_r) of (non-zero) Laurent polynomials $f_i \in k[\mathbf{X}^{\pm 1}]$; examples of such integrals are e.g. provided by Theorem 3.3. Given $\omega \in \mathbf{R}^n$ and $J \subset \{1, \dots, r\}$, let $V_{\omega, J}$ be the subvariety of \mathbf{T}_k^n defined by the vanishing of $\text{in}_\omega(f_j)$ for all $j \in J$ and the non-vanishing of $\prod_{i \notin J} \text{in}_\omega(f_i)$. Then, assuming non-degeneracy of (f_1, \dots, f_r) , the varieties V_i in a formula (4.1) produced using [35–37] are all of the form $V_{\omega, J}$ and hence satisfy Assumption 4.4.

5. Counting rational points on subvarieties of tori

Assuming the validity of Assumption 4.4, this section is devoted to “computing” the numbers $\#\bar{V}_i(\mathfrak{O}_K/\mathfrak{P}_K)$ in (4.1). Using the inclusion-exclusion principle, we may reduce to the case that each V_i is a *closed* subvariety of an algebraic torus $\mathbf{T}_k^{n_i}$. Note that the non-constructive version of the Uniformity Problem from §1 has a positive solution whenever each $\#\bar{V}_i(\mathfrak{O}_K/\mathfrak{P}_K)$ is a polynomial in q_K (after excluding finitely many places of k). The following method is based on the heuristic observation that the latter condition is often satisfied for examples of interest.

Setup. As in §4.3, we write $\mathbf{T}^n = \text{Spec}(\mathbf{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$ and $\mathbf{T}_R^n = \mathbf{T}^n \otimes R$. For a finite set $S \subset \mathcal{V}_k$, let $\mathfrak{o}_S := \{x \in k : x \in \mathfrak{o}_v \text{ for all } v \in \mathcal{V}_k \setminus S\}$ denote the usual ring of S -integers of k . For $f_1, \dots, f_r \in \mathfrak{o}_S[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, define

$$(f_1, \dots, f_r)_S^n := \text{Spec}(\mathfrak{o}_S[X_1^{\pm 1}, \dots, X_n^{\pm 1}]/\langle f_1, \dots, f_r \rangle) \subset \mathbf{T}_{\mathfrak{o}_S}^n.$$

For $v \in \mathcal{V}_k \setminus S$ and a finite extension \mathfrak{K} of $\mathfrak{o}/\mathfrak{p}_v$, let $|f_1, \dots, f_r|_{\mathfrak{K}}^n$ denote the number of \mathfrak{K} -rational points of $(f_1, \dots, f_r)_S^n$.

Objective: symbolic enumeration. From now on, let $f_1, \dots, f_r \in \mathfrak{o}_S[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be given as above. Our goal in the following is to symbolically “compute” the numbers $|f_1, \dots, f_r|_{\mathfrak{K}}^n$ as a function of \mathfrak{K} . More precisely, the procedure described below constructs a polynomial, $H(X, c_1, \dots, c_\ell)$ say, over \mathbf{Z} such that, after possibly enlarging S , for all $v \in \mathcal{V}_k \setminus S$ and all finite extensions \mathfrak{K} of $\mathfrak{o}/\mathfrak{p}_v$,

$$(f_1, \dots, f_r)_{\mathfrak{K}}^n = H(|\mathfrak{K}|, \#\mathbf{U}_1(\mathfrak{K}), \dots, \#\mathbf{U}_\ell(\mathfrak{K})),$$

where each \mathbf{U}_i is an explicitly given closed subscheme of some $\mathbf{T}_{\mathfrak{o}_S}^{n_i}$. We could of course simply take $H = c_1$ and $\mathbf{U}_1 = (f_1, \dots, f_r)_S^n$ but we seek to do better. Indeed, in many cases of interest, H can be taken to be a polynomial in X only. In the following, we describe a method which has proven to be quite useful for handling such cases.

Dimension ≤ 1 . We first describe two base cases of our method. Namely, if $n = 0$, then, after possibly enlarging S , $(f_1, \dots, f_r)_S^n$ is either \emptyset or $\mathbf{T}_{\mathfrak{o}_S}^0 = \text{Spec}(\mathfrak{o}_S)$ depending on whether some $f_i \neq 0$ or not; thus, $|f_1, \dots, f_r|_{\mathfrak{K}}^n \in \{0, 1\}$ for \mathfrak{K} as above.

Secondly, if $n = 1$, then we use the Euclidean algorithm over k (thus possibly enlarging S) to compute a single square-free polynomial $f \in \mathfrak{o}_S[X_1]$ such that $(f_1, \dots, f_r)_S^1 = (f)_S^1$. If f splits completely over k , then, after possibly enlarging S once again, $|f|_{\mathfrak{K}}^1 = \deg(f)$ for all \mathfrak{K} as above. If f does not split completely over k , then we introduce a new variable, c_f say, corresponding to the number of solutions of $f = 0$ in \mathfrak{K}^\times .

Simplification. It is often useful to “simplify” the given Laurent polynomials f_1, \dots, f_r ; while this step was sketched in [36, §6.6], here we provide some further details. As before, the set S may need to be enlarged at various points in the following. First, we

discard any zero polynomials among the f_i . We then clear denominators so that each $f_i \in k[X_1, \dots, X_n]$ is an actual (not just Laurent) polynomial. Next, we replace each f_i by its square-free part in $k[X_1, \dots, X_n]$. For each pair (i, j) of distinct indices, we then compute the (square-free part of the) remainder, r say, of f_i after multivariate polynomial division by f_j with respect to some term order (see e.g. [1, §1.5]). If r consists of fewer terms than f_i , we replace f_i by r . Next, for each pair (i, j) as above and each term t_i of f_i and t_j of f_j , we are free to replace f_i by (the square-free part of) $\frac{t_j}{g} f_i - \frac{t_i}{g} f_j$, where $g = \gcd(t_i, t_j)$ (computed over k), which we again do whenever it reduces the total number of terms. After finitely many iterations of the above steps, f_1, \dots, f_r will stabilise at which point we conclude the simplification step.

We next describe two procedures which, if applicable, allow us to express $|f_1, \dots, f_r|_{\mathfrak{K}}^n$ in terms of the numbers of rational points of subschemes of lower-dimensional tori. We then recursively attempt to solve the symbolic enumeration problem from above for these.

Reduction of dimension I: torus factors. As explained in [36, §6.3], using the natural action of $\mathrm{GL}_n(\mathbf{Z})$ on \mathbf{T}^n , a Smith normal form computation allows us to effectively construct $g_1, \dots, g_r \in \mathfrak{o}_S[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ and an explicit isomorphism $(f_1, \dots, f_r)_S^n \approx (g_1, \dots, g_r)_S^d \times_{\mathfrak{o}_S} \mathbf{T}_{\mathfrak{o}_S}^{n-d}$, where d is the dimension of the Newton polytope of $f_1 \cdots f_r$ (see §4.3). It follows that for all \mathfrak{K} as above, $|f_1, \dots, f_r|_{\mathfrak{K}}^n = |g_1, \dots, g_r|_{\mathfrak{K}}^d \cdot (|\mathfrak{K}| - 1)^{n-d}$. In the following, we may thus assume that $n = d$.

Reduction of dimension II: solving for variables. Whenever it is applicable (after first suitably permuting the variables, if required), the following lemma allows us to replace the problem of symbolically computing $|f_1, \dots, f_r|_{\mathfrak{K}}^n$ by four instances of the same problem in dimension $n - 1$.

Lemma 5.1. *Let $F \subset \mathfrak{o}_S[X_1^{\pm 1}, \dots, X_{n-1}^{\pm 1}]$. Further let $f = u - wX_n$ for non-zero $u, w \in \mathfrak{o}_S[X_1^{\pm 1}, \dots, X_{n-1}^{\pm 1}]$. Then for all $v \in \mathcal{V}_k \setminus S$ and all finite extensions \mathfrak{K} of $\mathfrak{o}/\mathfrak{p}_v$,*

$$|F, f|_{\mathfrak{K}}^n = |F|_{\mathfrak{K}}^{n-1} - |F, u|_{\mathfrak{K}}^{n-1} - |F, w|_{\mathfrak{K}}^{n-1} + |\mathfrak{K}| \cdot |F, u, w|_{\mathfrak{K}}^{n-1}.$$

Proof. Projection onto the first $n - 1$ coordinates induces an isomorphism of \mathfrak{o}_S -schemes $(F, f)_S^n \setminus (F, f, w)_S^n \approx (F)_S^{n-1} \setminus (F, uw)_S^{n-1}$. As $(F, f, w)_S^n = (F, u, w)_S^n \approx (F, u, w)_S^{n-1} \times_{\mathfrak{o}_S} \mathbf{T}_{\mathfrak{o}_S}^1$, the claim follows since for all $v \in \mathcal{V}_k \setminus S$ and all finite extensions \mathfrak{K} of $\mathfrak{o}/\mathfrak{p}_v$,

$$|((F)_S^{n-1} \setminus (F, uw)_S^{n-1})(\mathfrak{K})| = |F|_{\mathfrak{K}}^{n-1} - |F, u|_{\mathfrak{K}}^{n-1} - |F, w|_{\mathfrak{K}}^{n-1} + |F, u, w|_{\mathfrak{K}}^{n-1}. \quad \blacklozenge$$

The evident analogue of Lemma 5.1 for Euler characteristics of closed subvarieties of \mathbf{T}_k^n has already been used in the author's software package Zeta [40] for computing topological zeta functions. The same is true of the following straightforward generalisation of the special case that $w \in \mathfrak{o}_S^\times$ in Lemma 5.1 (cf. [36, §6.6]).

Lemma 5.2. *Let $F \subset \mathfrak{o}_S[X_1^{\pm 1}, \dots, X_{n-1}^{\pm 1}, X_n]$. Further let $f = u - wX_n$ for $u \in \mathfrak{o}_S[X_1^{\pm 1}, \dots, X_{n-1}^{\pm 1}]$ and $w \in \mathfrak{o}_S^\times$. Let $\tilde{F} := \{f(X_1, \dots, X_{n-1}, w^{-1}u) : f \in F\}$. Then for all $v \in \mathcal{V}_k \setminus S$ and all finite extensions \mathfrak{K} of $\mathfrak{o}/\mathfrak{p}_v$,*

$$|F, f|_{\mathfrak{K}}^n = |\tilde{F}|_{\mathfrak{K}}^{n-1} - |\tilde{F}, u|_{\mathfrak{K}}^{n-1}. \quad \blacklozenge$$

Final case. Finally, if none of the above techniques for computing or decomposing $(f_1, \dots, f_r)_S^n$ applies, then we introduce a new variable corresponding to $|f_1, \dots, f_r|_{\mathfrak{K}}^n$. In order to avoid this step whenever possibly, we first attempt to apply the above steps (including all possible applications of Lemmas 5.1–5.2) without ever invoking this final case.

Example 5.3. We illustrate the symbolic enumeration of rational points using an example which features in the author's calculation of the generic local subalgebra zeta functions associated with $\mathfrak{gl}_2(\mathbf{Q})$ (see §9.1). Let $R := \mathbf{Z}[1/2]$ and let $V \subset \mathbf{T}_R^{10}$ be defined by $f = 0$, where

$$f := 2X_4X_5^2X_8 - 2X_2X_5X_7X_8 + X_1X_6X_7X_8 - 4X_3X_5^2X_9 + 4X_2X_5X_6X_9 - X_1X_6^2X_9.$$

We will now compute $\#V(\mathbf{F}_q)$, where $q > 1$ is an arbitrary odd prime power, as a polynomial in q . Note that X_{10} does not appear in f so that $\#V(\mathbf{F}_q)$ is certainly divisible by $q - 1$. Less obviously, one can check (e.g. using Sage [13]) that the Newton polytope of f has dimension 4 whence V admits \mathbf{T}_R^6 as a direct factor. Indeed, by performing a change of coordinates using the automorphism of $R[X_1^{\pm 1}, \dots, X_{10}^{\pm 1}]$ given by

$$\begin{aligned} X_1 &\mapsto X_1X_6X_7^{-1}X_8^{-2}X_9^2X_{10}, & X_5 &\mapsto X_9, & X_8 &\mapsto X_7, \\ X_2 &\mapsto X_2X_9X_8^{-1}X_{10}, & X_6 &\mapsto X_4X_6^{-1}X_7X_8, & X_9 &\mapsto X_6, \\ X_3 &\mapsto X_3X_6^{-1}X_7X_{10}, & X_7 &\mapsto X_8, & X_{10} &\mapsto X_5, \\ X_4 &\mapsto X_{10}, \end{aligned}$$

the polynomial f is transformed into $X_7X_9^2X_{10} \cdot f'(X_1, X_2, X_4, X_3)^1$, where

$$f' = -X_1X_3^2 + X_1X_3 + 4X_2X_3 - 2X_2 - 4X_4 + 2 \in R[X_1^{\pm 1}, \dots, X_4^{\pm 1}].$$

Hence, if $V' \subset \mathbf{T}_R^4$ is defined by $f' = 0$, then $V \approx \mathbf{T}_R^6 \times_R V'$.

Regarding V' , we observe that $f' = 0$ is equivalent to $X_4 = g(X_1, X_3, X_2)/4$, where

$$g = -X_1X_2^2 + X_1X_2 + 4X_2X_3 - 2X_3 + 2.$$

Lemma 5.2 thus shows that for all odd prime powers $q > 1$, $\#V'(\mathbf{F}_q) = (q-1)^3 - \#V''(\mathbf{F}_q)$, where $V'' \subset \mathbf{T}_R^3$ is defined by $g = 0$.

In order to determine $\#V''(\mathbf{F}_q)$, we write $g = u - wX_3$, where $u := -X_1X_2^2 + X_1X_2 + 2$ and $w := 2 - 4X_2$, and invoke Lemma 5.1. We thus find that for odd prime powers $q > 1$,

$$\#V''(\mathbf{F}_q) = (q-1)^2 - \#A(\mathbf{F}_q) - \#B(\mathbf{F}_q) + q \cdot \#C(\mathbf{F}_q),$$

¹The variables are renumbered in order to be able to invoke Lemmas 5.1–5.2 verbatim later on.

where A , B , and C are the subschemes of \mathbf{T}_R^2 defined by $u = 0$, $w = 0$, and $u = w = 0$, respectively. By rewriting $w = 0$ as $X_2 = 1/2$, Lemma 5.2 shows that $\#B(\mathbf{F}_q) = q - 1$. Moreover, $\#C(\mathbf{F}_q) = \#D(\mathbf{F}_q)$, where $D \subset \mathbf{T}_R^1$ is defined by $0 = u(X_1, 1/2) = X_1/4 + 2$, so that $\#C(\mathbf{F}_q) = 1$.

Finally, in order to compute $\#A(\mathbf{F}_q)$, we write $u = u' - w'(X_2)X_1$, where $u' := 2$ and $w'(X_2) := X_2^2 - X_2$, and use Lemma 5.1 once more. Since the numbers of \mathbf{F}_q -rational points of the subschemes of \mathbf{T}_R^1 defined by $u' = 0$ and $w'(X_1) = 0$ are 0 and 1, respectively, we thus find that $\#A(\mathbf{F}_q) = q - 2$.

In summary, if $q > 1$ is an odd prime power, then

$$\begin{aligned} \#V(\mathbf{F}_q) &= (q - 1)^6 \cdot \#V'(\mathbf{F}_q) = (q - 1)^6 \cdot ((q - 1)^3 - \#V''(\mathbf{F}_q)) \\ &= (q - 1)^6 \cdot \left((q - 1)^3 - ((q - 1)^2 - \#A(\mathbf{F}_q) - \#B(\mathbf{F}_q) + q \cdot \#C(\mathbf{F}_q)) \right) \\ &= (q - 1)^6 \cdot \left((q - 1)^3 - ((q - 1)^2 - (q - 2) - (q - 1) + q) \right) \\ &= (q - 1)^6 \cdot (q^3 - 4q^2 + 6q - 5). \end{aligned}$$

6. Local zeta functions as sums of rational functions

Suppose that Assumptions 4.2, 4.4, and 4.5 are satisfied. Our first task in this section is to rewrite (4.1) as a sum of explicitly given rational functions. With the method from §5 at our disposal, this problem reduces to finding such an expression for each W_i . We will see that Barvinok's algorithm from convex geometry solves this problem. Our second task then concludes the computation of the generic local zeta functions in Theorem 4.1; it is concerned with adding a potentially large number of multivariate rational functions. We describe a method aimed towards improving the practicality of this step which, while mathematically trivial, often vastly dominates the run-time of our computations.

6.1. Barvinok's algorithm: generating functions and substitutions

Let $\mathcal{P} \subset \mathbf{R}_{\geq 0}^n$ be a rational polyhedron and let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be algebraically independent over \mathbf{Q} . It is well-known that the generating function $|\mathcal{P}| := \sum_{\alpha \in \mathcal{P} \cap \mathbf{Z}^n} \boldsymbol{\lambda}^\alpha$ is rational in the sense that within the field of fractions of $\mathbf{Q}[[\lambda_1, \dots, \lambda_n]]$, it belongs to $\mathbf{Q}(\lambda_1, \dots, \lambda_n)$. The standard proof of this fact (see e.g. [4, Ch. 13]) proceeds by reducing to the case that \mathcal{P} is a cone, in which case an explicit formula for $|\mathcal{P}|$ can be derived from a triangulation of \mathcal{P} via the inclusion-exclusion principle. This strategy for computing $|\mathcal{P}|$ is, however, of rather limited practical use.

A far more sophisticated approach is given by ‘‘Barvinok's algorithm’’; see [5, 7] for details and [49] for a high-level overview. Barvinok's algorithm computes $|\mathcal{P}|$ for each (suitably encoded) rational polyhedron $\mathcal{P} \subset \mathbf{R}_{\geq 0}^n$ as a sum of rational functions of the form $c\boldsymbol{\lambda}^{\alpha_0} / ((1 - \boldsymbol{\lambda}^{\alpha_1}) \cdots (1 - \boldsymbol{\lambda}^{\alpha_n}))$ for suitable $\alpha_0, \dots, \alpha_n \in \mathbf{Z}^n$ and $c \in \mathbf{Q}$. For a fixed ambient dimension $n \geq 1$, his algorithm runs in polynomial time so that $|\mathcal{P}|$ is computed as a short sum of short rational functions in a precise technical sense. Beyond

its theoretical strength, Barvinok’s algorithm is also powerful in practice as demonstrated by the software implementation `LattE` [3].

In the setting of Assumption 4.5, we are not primarily interested in generating functions associated with polyhedra themselves but in rational functions derived from such generating functions via monomial substitutions. In detail, let $\xi = (\xi_1, \dots, \xi_m)$ be algebraically independent over \mathbf{Q} and let $\sigma_1, \dots, \sigma_n \in \mathbf{Z}^m$. Suppose that $\mathcal{P} \subset \mathbf{R}_{\geq 0}^n$ is a rational polyhedron such that $W := |\mathcal{P}|(\xi^{\sigma_1}, \dots, \xi^{\sigma_n})$ is well-defined on the level of rational functions. In principle, we could compute W by first using the output of Barvinok’s algorithm in order to write $|\mathcal{P}|$ in lowest terms, followed by an application of the given substitution. This method is, however, often impractical due to the computational cost of (multivariate) rational function arithmetic.

A theoretically favourable and also practical alternative is developed in [6, §2] (cf. [5, §5]). There, a polynomial time algorithm is described which takes as input a short representation of $|\mathcal{P}|$ (as, in particular, provided by Barvinok’s algorithm) and constructs a similar short representation for W . The important point to note here is that while we assumed the substitution $\lambda_i \mapsto \xi^{\sigma_i}$ to be valid for $|\mathcal{P}|$ itself, it may be undefined for some of the summands in the expression provided by Barvinok’s algorithm.

6.2. Computing the W_i in (4.1)

We may now clarify the vague formulation of Assumption 4.5. Namely, up to a factor $(X-1)^a X^b$, the W_i in (4.1) are obtained by applying suitable monomial substitutions (see [35, Rem. 4.12] and [37, Thm 5.5]) to rational functions of the form $\mathcal{Z}^{\mathcal{C}_0, \mathcal{P}_1, \dots, \mathcal{P}_m}(\xi_0, \dots, \xi_m)$ from [35, Def. 3.6]. The latter functions can, by their definitions, be written as sums of rational functions obtained by applying suitable monomial substitutions to generating functions enumerating lattice points inside rational half-open cones; as explained in [36, §8.4], we may replace these half-open cones by rational polyhedra. We may thus use Barvinok’s algorithm as well as the techniques for efficient monomial substitutions from [6, §2] in order to write each W_i as a sum of bivariate rational functions of the form

$$f(X, Y) / ((1 - X^{a_1} Y^{b_1}) \dots (1 - X^{a_m} Y^{b_m})) \tag{6.1}$$

for suitable integers $a_i, b_i \in \mathbf{Z}$, $m \geq 0$, and $f(X, Y) \in \mathbf{Q}[X, Y]$.

6.3. Final summation

In the following, we allow $f(X, Y)$ in (6.1) to be an element of $\mathbf{Q}[X, Y, c_1, c_2, \dots]$. By taking into account the polynomials obtained using §5, at this point, we may thus assume that we constructed a finite sum of expressions (6.1) such that, after excluding finitely many places of k , the local zeta functions in Theorem 4.1 are obtained by specialising $X \mapsto q_K$, $Y \mapsto q_K^{-s}$, and $c_i \mapsto \#\mathbf{U}_i(\mathfrak{O}_K/\mathfrak{P}_K)$ for certain explicit subschemes \mathbf{U}_i of tori over \mathfrak{o} (or over \mathfrak{o}_S). All that remains to be done in order to recover the local zeta functions of interest is to write the given sum of expressions (6.1) in lowest terms.

While our intended applications of Barvinok’s algorithm lie well within the practical scope of `LattE` [3], it will often be infeasible to pass the rational functions (6.1) to a

computer algebra system in order carry out the final summation. In addition to the sheer number of rational functions to be considered, a key problem is due to the fact that the number of distinct pairs (a_i, b_i) arising from summands (6.1) often obscures the relatively simple shape of the final sum (i.e. the local zeta function to be computed). This is consistent with the well-known observation (see e.g. [11, §2.3]) that few candidate poles of local zeta functions as provided by explicit formulae (4.1) survive cancellation.

In order to carry out the final summation, we proceed in two stages. First, we use an idea due to Woodward [50, §2.5] and add and simplify those summands (6.1) such that distinguished pairs $1 - X^c Y^d$ occur in their written denominators; our hope here is that some rays (a_i, b_i) will be removed via cancellations. While this step is not essential, it might improve the performance and memory requirements of the final stage. Here, we first construct a common denominator of all the remaining rational functions (6.1). We then compute the final result by summing the (6.1) rewritten over our common denominator, followed by one final division. In addition to being trivially parallelisable, by only adding numerators, we largely avoid costly rational function arithmetic.

6.4. Implementation issues

The method for computing generic local subobject or representation zeta functions described above has been implemented (for $k = \mathbf{Q}$) by the author as part of his package `Zeta` [40] for Sage [13]. The program `LattE` [3] (which implements Barvinok’s algorithm) plays an indispensable role. Moreover, the computer algebra system `Singular` [25] features essentially in the initial stages of our method (as described in [36, 37]).

The author’s implementation is primarily designed to find instances of positive solutions to the Uniformity Problem; its functionality and practicality are both quite restricted in non-uniform cases. Furthermore, the author’s method supplements Woodward’s approach [50] for computing local (subalgebra and ideal) zeta functions as well as various ad hoc computations carried out by others without replacing them. In particular, various examples of local zeta functions computed by Woodward cannot be reproduced using the present method. In addition to the theoretical limitations of the techniques from [35–37], this is also partially due to practical obstructions: while some computations of topological zeta functions in [35–37] were already fairly involved, the present method is orders of magnitude more demanding.

6.5. An example: “graded” subalgebras of $\mathbf{Z}[T]/T^3$

Many of the techniques developed in the present article as well as many of those described in [35–37] can be traced back to the computation of the (previously known) subalgebra zeta function of $\mathfrak{sl}_2(\mathbf{Z}_p)$ for $p \neq 2$ in [35, §7.1]. Said computation, in particular, illustrates the overall structure of the present method while avoiding the more sophisticated ingredients. For instance, we may regard the ad hoc enumeration of rational points in [35, §7.1] as a simple application of §5. While the use of `LattE` is mentioned in [35, §7.1], subsequent computations with rational functions are not spelled out there as they are ill-suited for manual work; `Zeta` computes $\zeta_{\mathfrak{sl}_2(\mathbf{Z}_p)}^{\leq}(s)$ (for sufficiently large p) by adding about 400

rational functions in p and p^{-s} .

We now discuss a non-trivial example of an application of the method developed here in which the final summation of rational functions can be performed by hand. For $n \leq 4$, the subalgebra zeta function of $\mathbf{Z}_p[T]/T^n$ (for sufficiently large p) is given in Theorem 9.2 below. While the case $n = 2$ is essentially trivial, **Zeta** computes the formula for $n = 3$ by evaluating a sum of about 60 rational functions in p and p^{-s} . In order to obtain an easier calculation, we fix a suitable additive decomposition of $\mathbf{Z}[T]/T^3$ and consider the associated graded local subalgebra zeta functions.

Let t denote the image of T in $\mathbf{Z}[T]/T^3$. Define \mathbf{A} to be $\mathbf{Z}[T]/T^3$ together with the additive decomposition $\mathbf{A} = \langle 1, t \rangle \oplus \langle t^2 \rangle$; note that this decomposition is not a grading of the \mathbf{Z} -algebra \mathbf{A} in the usual sense. Using Theorem 3.3 (with $\mathfrak{D} = \mathbf{Z}_p$), for all primes p ,

$$\zeta_{\mathbf{A} \otimes \mathbf{Z}_p}^{\text{gr} \leq}(s) = (1 - p^{-1})^{-3} \int_{V_p} |x_1|^{s-1} |x_3|^{s-2} |x_4|^{s-1} d\mu(\mathbf{x}),$$

where $V_p = \{\mathbf{x} \in \mathbf{Z}_p^4 : x_3 \mid x_1 x_2, x_4 \mid x_2^2, x_4 \mid x_3^2\}$. It follows (see e.g. [35, Prop. 3.9]) that if \mathcal{C} denotes the polyhedral cone

$$\mathcal{C} = \{\alpha \in \mathbf{R}_{\geq 0}^4 : \alpha_3 \leq \alpha_1 + \alpha_2, \alpha_4 \leq 2\alpha_2, \alpha_4 \leq 2\alpha_3\},$$

then $\zeta_{\mathbf{A} \otimes \mathbf{Z}_p}^{\text{gr} \leq}(s)$ can be expressed in terms of the generating function $|\mathcal{C}| \in \mathbf{Q}(\lambda_1, \dots, \lambda_4)$ (see §6.1) as

$$\zeta_{\mathbf{A} \otimes \mathbf{Z}_p}^{\text{gr} \leq}(s) = (1 - p^{-1}) \cdot |\mathcal{C}|(p^{-s}, p^{-1}, p^{1-s}, p^{-s}). \quad (6.2)$$

Using Barvinok's algorithm via **LattE**, we obtain

$$\begin{aligned} |\mathcal{C}| = & \frac{1}{(1 - \lambda_3^{-1})(1 - \lambda_4^{-1})(1 - \lambda_2 \lambda_3 \lambda_4^2)(1 - \lambda_1 \lambda_3)} + \frac{1}{(1 - \lambda_1 \lambda_3)(1 - \lambda_2 \lambda_3)(1 - \lambda_3^{-1})(1 - \lambda_4)} \\ & - \frac{1}{(1 - \lambda_1)(1 - \lambda_2 \lambda_3 \lambda_4^2)(1 - \lambda_3^{-1})(1 - \lambda_4^{-1})} + \frac{1}{(1 - \lambda_1)(1 - \lambda_2 \lambda_3 \lambda_4^2)(1 - \lambda_3^{-1} \lambda_4^{-2})(1 - \lambda_4^{-1})} \\ & + \frac{1}{(1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3 \lambda_4^2)(1 - \lambda_4^{-1})} + \frac{1}{(1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)(1 - \lambda_4)}. \end{aligned} \quad (6.3)$$

Observe that while each summand in (6.3) is (up to a sign) the generating function associated with a (unimodular) cone, these cones are *not* contained in \mathcal{C} —this is typical for formulae obtained using Barvinok's algorithm.

In this specific example, the substitution in (6.2) can actually be applied to each of the summands in (6.3). The final summation is then easily carried out, even without the techniques from §6.3, and yields

$$\zeta_{\mathbf{A} \otimes \mathbf{Z}_p}^{\text{gr} \leq}(s) = \frac{1 - p^{-4s}}{(1 - p^{-s})^2 (1 - p^{-3s}) (1 - p^{1-2s})}.$$

7. Interlude: reduced representation zeta functions

Reduced zeta functions arising from the enumeration of subalgebras and ideals were introduced by Evseev [20]. They constitute a limit “ $p \rightarrow 1$ ” of suitable local zeta functions distinct from but related to the topological zeta functions of Denef and Loeser [12] (which were later adapted to the case of subobject zeta functions by du Sautoy and Loeser [16]). Informally, Evseev’s definition can be summarised as follows in our setting. Let \mathbf{A} be an \mathfrak{o} -form of a k -algebra as in §2.1. For each $v \in \mathcal{V}_k$, we may regard $\zeta_{\mathbf{A} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\leq}(s)$ as a (rational) formal power series in $Y = q_v^{-s}$. The reduced subalgebra zeta function of \mathbf{A} (an invariant of $\mathbf{A} \otimes_{\mathfrak{o}} \mathbf{C}$, in fact) is obtained by taking a limit “ $q_v \rightarrow 1$ ” applied to the coefficients of $\zeta_{\mathbf{A} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\leq}(s)$ as a series in Y . The rigorous definition of reduced zeta function in [20] involves the motivic subobject zeta functions introduced by du Sautoy and Loeser [16].

In this section, we show that “reduced representation zeta functions” associated with unipotent groups are always identically 1. In addition to imposing restrictions on the shapes of generic local representation zeta functions of such groups, this fact provides a simple necessary condition for the correctness of explicit calculations of local zeta functions such as those documented below.

We begin with a variation of a result from [39]. Let V be a separated k -scheme of finite type. For any embedding $k \subset \mathbf{C}$, the topological Euler characteristic $\chi(V(\mathbf{C}))$ is defined and well-known to be independent of the embedding; cf. [29].

Lemma 7.1. *Let V_1, \dots, V_r be separated \mathfrak{o} -schemes of finite type and $W_1, \dots, W_r \in \mathbf{Q}(X, Y_1, \dots, Y_m)$. Suppose that for almost all $v \in \mathcal{V}_k$ and all integers $f \geq 0$, each W_i is regular at (q_v^f, Y_1, \dots, Y_m) . Let $P \subset \mathcal{V}_k$ have natural density 1 and suppose that*

$$\sum_{i=1}^r \#V_i(\mathfrak{o}/\mathfrak{p}_v) \cdot W_i(q_v, Y_1, \dots, Y_m) = 0$$

for all $v \in P$. Then $\sum_{i=1}^r \chi(V_i(\mathbf{C})) \cdot W_i(1, Y_1, \dots, Y_m) = 0$.

Proof. Using [41, Ch. 4], in the setting of [39, Thm 3.7], we may assume that $\alpha(1_{\Gamma_S}) = \chi(V(\mathbf{C}))$. The claim is now an immediate consequence of [39, Thm 3.2] and its proof. \blacklozenge

Remark 7.2. Given a formula (4.1) for local subalgebra or ideal zeta functions such that the regularity conditions in Lemma 7.1 are satisfied, we may read off the associated reduced zeta function as $\sum_{i=1}^r \chi(V_i(\mathbf{C})) \cdot W_i(1, Y)$ without using motivic zeta functions.

The following is a consequence of the explicit formulae in [19].

Theorem 7.3. *Let \mathbf{G} be a unipotent algebraic group over k . Let \mathbf{G} be an \mathfrak{o} -form of \mathbf{G} as an affine group scheme of finite type. There are separated \mathfrak{o} -schemes U_1, \dots, U_ℓ of finite type and rational functions $W_1, \dots, W_\ell \in \mathbf{Q}(X, Y)$ such that*

$$(i) \text{ for almost all } v \in \mathcal{V}_k, \zeta_{\mathbf{G}(\mathfrak{o}_v)}^{\text{irr}}(s) - 1 = \sum_{i=1}^{\ell} \#U_i(\mathfrak{o}/\mathfrak{p}_v) \cdot W_i(q_v, q_v^{-s}),$$

- (ii) each W_i is regular at each point (q, Y) for $q \geq 1$, and
- (iii) $W_i(1, Y) = 0$ for $i = 1, \dots, \ell$.

Proof. We adopt the notation of [19]. By [19, Prop. 2.2 and 3.4], for almost all $v \in \mathcal{V}_k$,

$$\zeta_{\mathbf{G}(\mathfrak{o}_v)}^{\text{irr}}(s) - 1 = \sum_{i \in W'} c_i(\mathfrak{o}/\mathfrak{p}_v) \cdot W_i(q_v, q_v^{-s}),$$

where

$$W_i(X, Y) := X^{|U_i| - \binom{d}{2}} \cdot \prod_{u=1}^{d-1} \frac{1 - X^{-1}}{1 - X^{-u}} \cdot (1 - X^{-1})^{|U_i|+1} \cdot \prod_{j \in M_i} \frac{X^{-B_j} Y^{A_j}}{1 - X^{-B_j} Y^{A_j}}. \quad (7.1)$$

The numbers $c_i(\mathfrak{o}/\mathfrak{p}_v)$ are already of the form $\#\mathbf{U}_i(\mathfrak{o}/\mathfrak{p}_v)$ for suitable \mathbf{U}_i (cf. [19, §3.1]). Next, we may assume that the rational numbers A_j and B_j in [19, Prop. 3.4] and in (7.1) are actually integers so that $W_i(X, Y) \in \mathbf{Q}(X, Y)$; this follows e.g. by taking square roots of principal minors and rewriting the integral [19, (2.3)] similarly to [37, (4.3)] (so that, in particular, the fraction “ $-s/2$ ” in [19, (2.2)] is replaced by “ $-s$ ”).

Clearly, each $W_i(X, Y)$ is regular at each point (q, Y) with $q > 1$. Moreover, it follows from the definitions of M_i and U_i in [19, §3.1] that $|M_i| \leq |U_i| + 1$ for each $i \in W'$. As each $(1 - X^{-1})/(1 - X^{-u})$ and $(1 - X^{-1})/(1 - X^{-B_j} Y^{A_j})$ is regular at $(1, Y)$, the same is true of each $W_i(X, Y)$. Moreover, $(1 - X^{-1})/(1 - X^{-B_j} Y^{A_j})$ vanishes at $(1, Y)$ whenever $A_j \neq 0$. Finally, by [19, Rem. 3.6], for each $i \in W'$, there exists $j_0 \in M_i$ with $A_{j_0} > 0$ whence $W_i(1, Y) = 0$. \blacklozenge

Remark 7.4. Theorem 7.3 refines the simple observation that for almost all $v \in \mathcal{V}_k$, the coefficients of $\zeta_{\mathbf{G}(\mathfrak{o}_v)}^{\text{irr}}(s) - 1$ as a series in q_v^{-s} are non-negative integers divisible by $q_v - 1$, a simple consequence of the Kirillov orbit method. (Indeed, $(\mathfrak{o}/\mathfrak{p}_v)^\times$ acts freely on non-trivial characters while preserving the two types of radicals in [44, Thm 2.6].)

By combining Lemma 7.1 and Theorem 7.3, we obtain the following.

Corollary 7.5. *Let \mathbf{G} be as in Theorem 7.3. Let $\mathbf{V}_1, \dots, \mathbf{V}_r$ be separated \mathfrak{o} -schemes of finite type and let $W_1, \dots, W_r \in \mathbf{Q}(X, Y)$ such that for almost all $v \in \mathcal{V}_k$,*

$$\zeta_{\mathbf{G}(\mathfrak{o}_v)}^{\text{irr}}(s) = \sum_{i=1}^r \#\mathbf{V}_i(\mathfrak{o}/\mathfrak{p}_v) \cdot W_i(q_v, q_v^{-s}).$$

If each W_i is regular at (q, Y) for each $q \geq 1$, then $\sum_{i=1}^r \chi(\mathbf{V}_i(\mathbf{C})) \cdot W_i(1, Y) = 1$. \blacklozenge

Corollary 7.6. *Let \mathbf{G} be as in Theorem 7.3. Let $W(X, Y) \in \mathbf{Q}(X, Y)$ such that*

- (i) $W(X, Y)$ can be written over a denominator which is a product of non-zero factors of the form $1 - X^a Y^b$ for integers $a \geq 0$ and $b \geq 1$ and
- (ii) $\zeta_{\mathbf{G}(\mathfrak{o}_v)}^{\text{irr}}(s) = W(q_v, q_v^{-s})$ for almost all $v \in \mathcal{V}_k$.

Then $W(1, Y) = 1$. ◆

The assumptions in Corollary 7.6 are satisfied for many examples of interest; see Table 1. In fact, even the following much stronger assumptions are often satisfied.

Corollary 7.7. *Let \mathbf{G} be as in Theorem 7.3. Suppose that there are integers $a_i \geq 0$, $b_i \geq 1$, and $\varepsilon_i \in \{\pm 1\}$ for $i = 1, \dots, m$ such that for almost all $v \in \mathcal{V}_k$,*

$$\zeta_{\mathbf{G}(\mathfrak{o}_v)}^{\text{irr}}(s) = \prod_{i=1}^m (1 - q_v^{a_i - b_i s})^{\varepsilon_i}.$$

Then $\sum_{i=1}^m \varepsilon_i = 0$ and the multisets $\{b_i : \varepsilon_i = 1\}$ and $\{b_i : \varepsilon_i = -1\}$ coincide.

Proof. Corollary 7.6 shows that $1 = \prod_{i=1}^m (1 - Y^{b_i})^{\varepsilon_i}$. By considering the vanishing order of this function in Y at 1, we see that $\sum_{i=1}^m \varepsilon_i = 0$. Let $c = \max(b_i : \varepsilon_i = 1)$ and $d = \max(b_i : \varepsilon_i = -1)$. If $\xi \in \mathbf{C}$ is a primitive c th root of unity, then $1 - \xi^{b_i} = 0$ for some i with $\varepsilon_i = -1$ whence $c \leq b_i \leq d$; dually, $d \leq c$ and the final claim follows by induction. ◆

Remark 7.8. The above results carry over verbatim to the case of representation zeta functions of “principal congruence subgroups” $\mathbf{G}^m(\mathfrak{o}_v) := \exp(\mathfrak{p}_v^m \mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v)$ attached to an \mathfrak{o} -form of a perfect Lie k -algebra in [2]. For example, by [2, Thm E], the ordinary representation zeta function of $\text{SL}_3^1(\mathbf{Z}_p)$ ($p \neq 3$) is $W(p, p^{-s})$ for

$$W(X, Y) = \frac{(X^2 Y^2 + XY^2 + Y^3 + X^2 + XY + Y) \times (X^2 - Y)(X - Y)X^3}{(1 - X^2 Y^3)(1 - XY^2)}$$

and indeed $W(1, Y) = 1$.

8. Applications I: representation zeta functions of unipotent groups

In this and the following two sections, we record explicit examples of generic local zeta functions of groups, algebras, and modules of interest which were computed using the method developed in the present article and its implementation `Zeta` [40]. The explicit formulae given below, as well as others, are also included with `Zeta`.

It is well-known that, up to isomorphism, unipotent algebraic groups over k correspond 1–1 to finite-dimensional nilpotent Lie k -algebras, see [10, Ch. IV]. Nilpotent Lie algebras of dimension at most 6 over any field of characteristic zero were first classified by Morozov [32]; various alternative versions of this classification have been obtained.

As one of the main applications of the techniques developed in the present article, for an arbitrary number field k , we can compute the generic local (twist) representation zeta functions associated with all unipotent algebraic groups of dimension at most 6

over k . The results of these computations are documented in Table 1 (p. 24). The structure of Table 1 mimics the list of associated topological representation zeta functions in [37, Table 1]. In detail, the first column lists the relevant Lie algebras using de Graaf's notation [9]; an algebra $L_{d,i}$ has dimension d . For each Lie algebra \mathfrak{g} , we choose an \mathfrak{o} -form \mathbf{G} of the unipotent algebraic group over k associated with \mathfrak{g} . The second column in Table 1 contains formulae for the representation zeta functions of the groups $\mathbf{G}(\mathfrak{o}_v)$ which are valid for almost all $v \in \mathcal{V}_k$ (depending on \mathbf{G}). Note that Corollary 7.7 applies to the majority of examples in Table 1. As we previously documented in [37, §6], generic local representation zeta functions associated with various Lie algebras in Table 1 were previously known (but sometimes only recorded for $k = \mathbf{Q}$), as indicated in the third column. For the convenience of the reader, the more detailed references to the literature from [37, Tab. 1] are reproduced in Remark 8.3.

Remark 8.1 (From \mathbf{Q} to k). Apart from the four infinite families (see the following remark), all Lie algebras in Table 1 are defined over \mathbf{Q} . By the invariance of (4.1) under local base extensions (Theorem 4.1), it thus suffices to compute associated generic local representation zeta functions for $k = \mathbf{Q}$.

Remark 8.2 (Computations for infinite families). The method for computing generic local zeta functions developed in this article takes as input a global object such as a nilpotent Lie k -algebra. In order to carry out computations for the four infinite families $L_{6,19}(a)$, $L_{6,21}(a)$, $L_{6,22}(a)$, and $L_{6,24}(a)$ in Table 1, additional arguments are required.

First, as explained in [9], we are free to multiply the parameters a from above by elements of $(k^\times)^2 \leq k^\times$ without changing the k -isomorphism type of the Lie algebra, $\mathfrak{g}(a)$ say, in question. We may thus assume that $0 \neq a \in \mathfrak{o}$ in the following. The definition of $\mathfrak{g}(a)$ in [9] then provides us with a canonical \mathfrak{o} -form, $\mathfrak{g}(a)$ say, of $\mathfrak{g}(a)$ which is in fact defined over $\mathbf{Z}[a]$. Let \mathbf{G}_a be an \mathfrak{o} -form of the unipotent algebraic group over k associated with $\mathfrak{g}(a)$. As explained in [44, §2], the structure constants of $\mathfrak{g}(a)$ (with respect to its defining basis from [9]) give rise to a formula for $\zeta_{\mathbf{G}_a(\mathfrak{o}_v)}^{\text{irr}}(s)$ in terms of certain explicit \mathfrak{o} -defined p -adic integrals (see [44, Cor. 2.11]); this formula is valid for almost all $v \in \mathcal{V}_k$.

It is an elementary exercise to verify that if $\mathfrak{g} = L_{6,19}(a)$ or $\mathfrak{g} = L_{6,21}(a)$, then the polynomials featuring in the aforementioned integral formulae for $\zeta_{\mathbf{G}_a(\mathfrak{o}_v)}^{\text{irr}}(s)$ are all monomials in a and the variables Y_1, \dots, Y_d (in the notation of [44, §2.2] and up to signs). It follows that up to excluding finitely many $v \in \mathcal{V}_k$, $\zeta_{\mathbf{G}_a(\mathfrak{o}_v)}^{\text{irr}}(s)$ does not depend on a . We may therefore simply carry out our calculation for $k = \mathbf{Q}$ and $a = 1$, say.

Let $\mathfrak{g}(a)$ be $L_{6,22}(a)$ or $L_{6,24}(a)$. Another simple calculation reveals that (again up to signs) a single non-monomial polynomial occurs in the associated integral formulae from above, namely $Y_1^2 - aY_2^2$. For any fixed a , by applying the procedure from [37, §5.4] as well as the steps described in the present article, we produce a rational function $W_a(X, Y, Z)$ such that $\zeta_{\mathbf{G}_a(\mathfrak{o}_v)}^{\text{irr}}(s) = W_a(q_v, q_v^{-s}, c_a(v))$ for almost all $v \in \mathcal{V}_k$, where $c_a(v)$ denotes the number of roots of $X^2 - a$ in $\mathfrak{o}/\mathfrak{p}_v$; it is well-known that if $a \notin (k^\times)^2$, then for almost all $v \in \mathcal{V}_k$, $c_a(v) = 0$ or $c_a(v) = 2$ according to whether \mathfrak{p}_v remains inert or splits in $k(\sqrt{a})$, respectively. The critical observation (which follows easily from [37, §5.4]) is

that $W := W_a$ is independent of a and also of k . We may thus compute W explicitly by e.g. taking $k = \mathbf{Q}$ and $a = 2$.

Remark 8.3. Explicit references for the known instances of generic local representation zeta functions in Table 1 are as follows (cf. [37, Tab. 1]):

algebra	reference	algebra	reference
$L_{3,2}$	[34, Thm 5])	$L_{4,3}$	M_3 [21, (4.2.24)]
$L_{5,4}$	B_4 [42, Ex. 6.3])	$L_{5,5}$	$G_{5,3}$ [21, Tab. 5.2])
$L_{5,7}$	M_4 [21, (4.2.24)]	$L_{5,8}$	$M_{3,3}$ [21, (5.3.7)]
$L_{5,9}$	$F_{3,2}$ [21, Tab. 5.2])		$= G_3$ [42, Ex. 6.2])
$L_{6,10}$	$G_{6,12}$ [21, Tab. 5.2])	$L_{6,18}$	M_5 [21, (4.2.24)]
$L_{6,19}(0)$	$G_{6,7}$ [21, Tab. 5.2])	$L_{6,19}(a)$ ($a \in k^\times$)	$G_{6,14}$ [21, Tab. 5.2]
$L_{6,22}(0)$	[42, Ex. 6.5]	$L_{6,22}(a)$ ($a \in k^\times \setminus (k^\times)^2$) [22]	
$L_{6,25}$	$M_{4,3}$ [21, (5.3.7)]	$L_{6,26}$	$F_{1,1}$ [44, Thm B].

The author would like to emphasise that all the formulae in Table 1 were obtained using the method developed here. In particular, our computations provide independent confirmation of the aforementioned (sometimes computer-assisted but predominantly manual and ad hoc) calculations found in the literature.

For an example in dimension > 6 , recall from §2.3 that U_n denotes the group scheme of upper unitriangular $n \times n$ -matrices. Using the notation from [9] as in Table 1, $U_3 \otimes \mathbf{Q}$ (the Heisenberg group) and $U_4 \otimes \mathbf{Q}$ are the unipotent algebraic groups over \mathbf{Q} associated with the Lie algebras $L_{3,2}$ and $L_{6,19}(-1)$, respectively. The following result obtained using the method from the present article illustrates that the simple shapes of the corresponding local representation zeta functions in Table 1 may mislead.

Theorem 8.4. *For almost all primes p and all finite extensions K/\mathbf{Q}_p ,*

$$\zeta_{U_5(\mathfrak{O}_K)}^{\text{irr}}(s) = W(q_K, q_K^{-s}),$$

where

$$\begin{aligned} W = & (X^{10}Y^{10} - X^9Y^9 - 2X^9Y^8 + X^9Y^7 + X^8Y^8 - X^7Y^7 - 2X^7Y^6 + X^7Y^5 \\ & + 6X^6Y^6 - 4X^5Y^6 - 4X^5Y^4 + 6X^4Y^4 + X^3Y^5 - 2X^3Y^4 - X^3Y^3 \\ & + X^2Y^2 + XY^3 - 2XY^2 - XY + 1) \times (1 - Y^3) \times (1 - Y) \\ & / ((1 - X^6Y^4)(1 - X^3Y^3)(1 - XY^3)(1 - X^2Y^2)(1 - X^2Y)^2). \end{aligned}$$

The topological representation zeta function of U_6 cannot be computed using [37]. Consequently, the corresponding local zeta functions cannot be computed using the method developed here.

Observe that the numerator of each $W(X, Y)$ in Table 1 is divisible by a polynomial of the form $1 - Y^e$. Experimental evidence provided by these examples and those in **Zeta** suggests that the following p -adic version of [37, Qu. 7.4] might have a positive answer.

Question 8.5. Let G be an \mathfrak{o} -form of a non-abelian unipotent algebraic group over k . Does the meromorphic continuation of $\zeta_{G(\mathfrak{o}_v)}^{\text{irr}}(s)$ always vanish at zero for almost all $v \in \mathcal{V}_k$?

Remark. By [24, Cor. 2], if p is odd, then the meromorphic continuation of the ordinary (= non-twisted) representation zeta function of a compact FAb p -adic analytic group vanishes at -2 .

Lie algebra	$W(X, Y)$ s.t. $\zeta_{G(\mathfrak{o}_v)}^{\text{irr}}(s) = W(q_v, q_v^{-s})$ for almost all $v \in \mathcal{V}_k$	known
abelian	1	✓
$L_{3,2}$	$(1 - Y)/(1 - XY)$	✓
$L_{4,3}$	$(1 - Y)^2/(1 - XY)^2$	✓
$L_{5,4}$	$(1 - Y^2)/(1 - XY^2)$	✓
$L_{5,5}$	$(1 - XY^2)(1 - Y)/((1 - X^2Y^2)(1 - XY))$	✓
$L_{5,6}$	$(1 - X^2Y^2)(1 - Y)^2/((1 - X^3Y^2)(1 - XY)^2)$	
$L_{5,7}$	$(1 - Y)^2/((1 - X^2Y)(1 - XY))$	✓
$L_{5,8}$	$(1 - Y)/(1 - X^2Y)$	✓
$L_{5,9}$	$(1 - Y)^2/((1 - X^2Y)(1 - XY))$	✓
$L_{6,10}$	$(1 - Y^2)(1 - Y)/((1 - XY^2)(1 - XY))$	✓
$L_{6,11}$	$\frac{(-X^3Y^4 + X^3Y^3 - 2X^2Y^3 + 3X^2Y^2 - 3XY^2 + 2XY - Y + 1)(1 - Y)}{(1 - X^4Y^3)(1 - X^2Y^2)}$	
$L_{6,12}$	$(1 - X^2Y^2)(1 - Y)^2/((1 - X^3Y^2)(1 - XY)^2)$	
$L_{6,13}$	$\frac{(X^4Y^6 + X^4Y^5 - X^3Y^4 - 2X^2Y^3 - XY^2 + Y + 1)(1 - Y)^2}{(1 - X^3Y^3)(1 - X^2Y^2)(1 - XY^2)(1 - XY)}$	
$L_{6,14}$	$\frac{(X^4Y^6 - X^4Y^4 + X^3Y^5 - 2X^2Y^3 + XY - Y^2 + 1)(1 - Y)^2}{(1 - X^3Y^3)(1 - X^3Y^2)(1 - XY^2)(1 - XY)}$	
$L_{6,15}$	$\frac{(-X^5Y^4 - X^4Y^3 + X^3Y^2 - X^2Y^2 + XY + 1)(1 - Y)^2}{(1 - X^5Y^3)(1 - X^3Y^2)(1 - XY)}$	
$L_{6,16}$	$(1 - Y^2)(1 - Y)^2/((1 - X^2Y)(1 - XY^2)(1 - XY))$	
$L_{6,17}$	$(1 - X^3Y^2)(1 - Y)^2/((1 - X^4Y^2)(1 - X^2Y)(1 - XY))$	
$L_{6,18}$	$(1 - Y)^2/((1 - X^3Y)(1 - XY))$	✓
$L_{6,19}(0)$	$(1 - Y)^2/((1 - X^2Y)(1 - XY))$	✓
$L_{6,19}(a)$ ($a \in k^\times$)	$(1 - Y^2)(1 - Y)/((1 - X^2Y)(1 - XY^2))$	✓
$L_{6,20}$	$(1 - XY^2)(1 - Y)/((1 - X^2Y^2)(1 - X^2Y))$	
$L_{6,21}(0)$	$(1 - Y)^2/(1 - X^2Y)^2$	
$L_{6,21}(a)$ ($a \in k^\times$)	$(1 - X^2Y^2)(1 - Y)^2/((1 - X^3Y^2)(1 - X^2Y)(1 - XY))$	
$L_{6,22}(0)$	$(1 - X^2Y^2)(1 - Y)/((1 - X^3Y^2)(1 - XY))$	✓
$L_{6,22}(a)$	if \mathfrak{p}_v splits in $k(\sqrt{a})$: $(1 - Y)^2/(1 - XY)^2$	✓
$(a \in k^\times \setminus (k^\times)^2)^2$	if \mathfrak{p}_v is inert in $k(\sqrt{a})$: $(1 - Y^2)/(1 - X^2Y^2)$	✓
$L_{6,23}$	$(1 - X^3Y^2)(1 - Y)/((1 - X^4Y^2)(1 - X^2Y))$	
$L_{6,24}(0)$	$\frac{(X^4Y^4 - X^4Y^3 + X^3Y^3 - 2X^2Y^2 + XY - Y + 1)(1 - Y)}{(1 - X^3Y^2)^2(1 - XY)}$	
$L_{6,24}(a)$	if $a \in (k^\times)^2$ or \mathfrak{p}_v splits in $k(\sqrt{a})$: $\frac{(-XY^2 + 2XY - 2Y + 1)(1 - Y)}{(1 - X^3Y^2)(1 - XY)}$	
$(a \in k^\times)$	if $a \notin (k^\times)^2$ and \mathfrak{p}_v is inert in $k(\sqrt{a})$: $\frac{(1 - XY^2)(1 - Y)}{(1 - X^3Y^2)(1 - XY)}$	
$L_{6,25}$	$(1 - XY)(1 - Y)/(1 - X^2Y)^2$	✓
$L_{6,26}$	$(1 - Y)/(1 - X^3Y)$	✓

Table 1: Generic local representation zeta functions associated with all indecomposable unipotent algebraic groups of dimension at most six over a number field

9. Applications II: classical subobject zeta functions

9.1. Subalgebras: $\mathfrak{gl}_2(\mathbf{Q})$

The first computations of the subalgebra zeta functions of $\mathfrak{sl}_2(\mathbf{Z}_p)$ are due, independently, to du Sautoy [14] (for $p \neq 2$, relying heavily on [28]) and White [48]. These zeta functions have later been confirmed by different means in [17], [30, §4.2], and [35, §7.1] (for $p \neq 2$). Up until now, $\mathfrak{sl}_2(\mathbf{Q})$ has remained the sole example of an insoluble Lie \mathbf{Q} -algebra whose generic local subalgebra zeta functions have been computed. Using the method developed in the present article, we obtain the following.

Theorem 9.1. *For almost all primes p and all finite extensions K/\mathbf{Q}_p ,*

$$\zeta_{\mathfrak{gl}_2(\mathfrak{O}_K)}^{\leq}(s) = W(q_K, q_K^{-s}),$$

where

$$\begin{aligned} W(X, Y) = & (-X^8Y^{10} - X^8Y^9 - X^7Y^9 - 2X^7Y^8 + X^7Y^7 - X^6Y^8 \\ & - X^6Y^7 + 2X^6Y^6 - 2X^5Y^7 + 2X^5Y^5 - 3X^4Y^6 + 3X^4Y^4 \\ & - 2X^3Y^5 + 2X^3Y^3 - 2X^2Y^4 + X^2Y^3 + X^2Y^2 - XY^3 \\ & + 2XY^2 + XY + Y + 1) / \left((1 - X^7Y^6)(1 - X^3Y^3)(1 - X^2Y^2)^2(1 - Y) \right). \end{aligned}$$

The topological subalgebra zeta function $\zeta_{\mathfrak{gl}_2(\mathbf{Q}), \text{top}}^{\leq}(s) = (27s - 14) / (6(6s - 7)(s - 1)^3s)$ of $\mathfrak{gl}_2(\mathbf{Q})$ was first recorded in [35, §7.3] (relying on techniques from [36]); the result given there is consistent with Theorem 9.1. Theorem 9.1 is particularly interesting since the simple shape of $\zeta_{\mathfrak{gl}_2(\mathbf{Q}), \text{top}}^{\leq}(s)$ might seem indicative of a local zeta function which is a product of ‘‘cyclotomic factors’’ $1 - q_K^{a-bs}$ or their inverses, which is in fact not the case.

We note that the computations underpinning Theorem 9.1 used that $\mathfrak{gl}_2(R) \approx \mathfrak{sl}_2(R) \oplus R$ for any commutative ring R in which 2 is invertible; here we regarded R as an abelian Lie R -algebra. Theorem 9.1 therefore also illustrates the potentially wild effect of direct sums on subalgebra zeta functions; in contrast, [18] contains examples of subalgebra and ideal zeta functions associated with nilpotent Lie algebras which are very well-behaved under this operation.

The rational function $W(X, Y)$ in Theorem 9.1 satisfies the functional equation

$$W(X^{-1}, Y^{-1}) = X^6Y^4W(X, Y)$$

predicted by [46, Thm A] (cf. [39, §5]). Moreover, the reduced subalgebra zeta function of $\mathfrak{gl}_2(\mathbf{Q})$ is $W(1, Y) = (1 - Y^3) / ((1 - Y)^3(1 - Y^2)^2)$, as predicted by [20, Thm 3.3] (using the fact that the reduced subalgebra zeta function of $\mathfrak{sl}_2(\mathbf{Z})$ is $(1 - Y^3) / ((1 - Y)^2(1 - Y^2)^2)$ (by [20, Prop. 4.1])).

²For $a \in (k^\times)^2$, $L_{6,22}(a) \approx L_{3,2}^2$ decomposes.

9.2. Subalgebras: $k[T]/T^n$ for $n \leq 4$

Most examples of local subalgebra zeta functions in the literature are concerned with (often nilpotent) Lie algebras. An important exception is given by the subalgebra zeta functions of \mathbf{Z}_p^n endowed with component-wise multiplication; explicit formulae for these zeta functions are known for $n \leq 3$ (see [33]). In the following, we consider another natural family of associative, commutative algebras, $k[T]/T^n$, for $n \leq 4$.

Due to the simplicity of the associated ‘‘cone integrals’’ as in [15], the formulae for $n = 2, 3$ recorded in the following can be obtained by hand with little difficulty. Using a substantially more involved computation, the techniques developed in the present article also allow us to consider the case $n = 4$. For $n = 5$, the author’s techniques for computing topological subalgebra zeta functions do not apply, i.e. Assumption 4.2 is violated.

Theorem 9.2. *For almost all primes p and all finite extensions K/\mathbf{Q}_p , writing $q = q_K$,*

$$\begin{aligned} \zeta_{\mathcal{D}_K[T]/T^2}^{\leq}(s) &= \frac{1 - q^{-2s}}{(1 - q^{-s})^2(1 - q^{1-2s})}, \\ \zeta_{\mathcal{D}_K[T]/T^3}^{\leq}(s) &= F_{\mathbf{Q}[T]/T^3}(q, q^{-s}) \times \frac{1 - q^{2-4s}}{(1 - q^{4-5s})(1 - q^{2-3s})^2(1 - q^{1-2s})(1 - q^{-s})}, \text{ and} \\ \zeta_{\mathcal{D}_K[T]/T^4}^{\leq}(s) &= F_{\mathbf{Q}[T]/T^4}(q, q^{-s}) / ((1 - q^{13-13s})(1 - q^{9-9s})(1 - q^{8-8s})(1 - q^{6-6s})^2 \\ &\quad \times (1 - q^{5-6s})(1 - q^{5-5s})(1 - q^{3-4s})(1 - q^{-s})), \end{aligned}$$

where $F_{\mathbf{Q}[T]/T^3} = -X^4Y^7 - X^4Y^6 - X^3Y^5 + X^3Y^4 - X^2Y^4 + X^2Y^3 - XY^3 + XY^2 + Y + 1$ and $F_{\mathbf{Q}[T]/T^4} = 1 + \dots - X^{49}Y^{54} \in \mathbf{Q}[X, Y]$ is given in Appendix A.

The topological subalgebra zeta function of $\mathbf{Q}[T]/T^4$ can be found in [36, §9.2]. As in §9.1, the zeta functions in Theorem 9.2 satisfy the functional equations predicted by [46, Thm A] and the associated reduced subalgebra zeta functions coincide with those computed using [20]; while Evseev only considered reduced zeta functions of *Lie* algebras, his reasoning also applies to more general, possibly non-associative, algebras. For example, using Theorem 9.2, after considerable cancellation, we find the reduced subalgebra zeta function of $\mathbf{Q}[T]/T^4$ to be $(Y^6 + Y^4 + 2Y^3 + Y^2 + 1) / ((1 - Y^6)(1 - Y^2)(1 - Y)^2)$, as predicted by Evseev’s results.

9.3. Subalgebras: soluble, non-nilpotent Lie algebras

Taylor [45, Ch. 6] computed local subalgebra zeta functions associated with soluble, non-nilpotent Lie algebras of the form $k^d \rtimes k$ (semidirect sum) for $d = 2, 3$, where k^d and k are regarded as abelian Lie algebras. In particular, he (implicitly) computed the subalgebra zeta function of the Lie algebra $\mathfrak{tt}_2(\mathbf{Z}_p)$ of upper triangular 2×2 -matrices over \mathbf{Z}_p (see [18, §3.4.2]). Klopsch and Voll [30] computed subalgebra zeta functions of arbitrary 3-dimensional Lie \mathbf{Z}_p -algebras in terms of Igusa’s local zeta functions attached to associated quadratic forms. Regarding the enumeration of ideals of soluble, non-nilpotent Lie algebras, Woodward [51] computed local ideal zeta functions of $\mathfrak{tt}_d(\mathbf{Z}_p)$ and certain combinatorially defined quotients of these algebras.

Since, to the author's knowledge, no examples of generic local subalgebra zeta functions associated with soluble, non-nilpotent Lie algebras of dimension 4 have been recorded in the literature, we now include some examples.

Theorem 9.3. *Let M^i denote an arbitrary but fixed \mathbf{Z} -form of the soluble Lie \mathbf{Q} -algebra M^i of dimension 4 from [8]. Then for almost all primes p and all finite extensions K/\mathbf{Q}_p , writing $q = q_K$,*

$$\begin{aligned}\zeta_{M_{0,0}^6 \otimes \mathfrak{D}_K}^{\leq} &= (q^{8-7s} - q^{7-5s} + q^{6-5s} - 2q^{5-4s} + q^{4-4s} + q^{4-3s} - 2q^{3-3s} + q^{2-2s} \\ &\quad - q^{1-2s} + 1) / ((1 - q^{6-4s})(1 - q^{3-2s})^2(1 - q^{1-s})^2(1 - q^{-s})), \\ \zeta_{M^8 \otimes \mathfrak{D}_K}^{\leq} &= (q^{5-7s} - 3q^{4-5s} + q^{4-4s} + 2q^{3-5s} - 2q^{3-4s} + q^{3-3s} + q^{2-4s} \\ &\quad - 2q^{2-3s} + 2q^{2-2s} + q^{1-3s} - 3q^{1-2s} + 1) \\ &\quad / ((1 - q^{6-5s})(1 - q^{2-2s})(1 - q^{1-s})^3(1 - q^{-s})), \\ \zeta_{M^{12} \otimes \mathfrak{D}_K}^{\leq} &= \frac{1 - q^{2-3s}}{(1 - q^{3-2s})(1 - q^{2-2s})(1 - q^{2-s})(1 - q^{1-s})(1 - q^{-s})}, \\ \zeta_{M_0^{13} \otimes \mathfrak{D}_K}^{\leq} &= \frac{-q^{4-5s} - q^{3-4s} + q^{3-3s} - 2q^{2-3s} + 2q^{2-2s} - q^{1-2s} + q^{1-s} + 1}{(1 - q^{4-3s})(1 - q^{3-2s})(1 - q^{2-2s})(1 - q^{1-s})(1 - q^{-s})}.\end{aligned}$$

Remark 9.4. Let \mathfrak{g} be the non-abelian Lie \mathbf{Q} -algebra of dimension 2. Define a \mathbf{Z} -form \mathfrak{g} of \mathfrak{g} by $\mathfrak{g} = \mathbf{Z}x \oplus \mathbf{Z}y$ and $[x, y] = y$. Then it is easy to see that for all p -adic fields K , $\zeta_{\mathfrak{g} \otimes \mathfrak{D}_K}^{\leq}(s) = 1 / ((1 - q_K^{-s})(1 - q_K^{1-s}))$. Using the notation from [8] as in Theorem 9.3, $M^8 \approx \mathfrak{g} \oplus \mathfrak{g}$ and $M_0^{13} \approx \mathfrak{g} \otimes_{\mathbf{Q}} \mathbf{Q}[X]/X^2$.

9.4. Submodules: U_n for $n \leq 5$ and relatives

For any commutative ring R , we consider

$$U_n(R) = \begin{bmatrix} 1 & R & \cdots & R \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

together with its natural action on R^n by right-multiplication. For $n \leq 4$, the determination of submodule zeta functions associated with U_n in the following is quite straightforward, even without the techniques developed here; the case $n = 5$, however, is rather more complicated, as is the resulting formula.

Theorem 9.5. For almost all primes p and all finite extensions K/\mathbf{Q}_p , writing $q = q_K$,

$$\begin{aligned}\zeta_{U_2(\mathfrak{O}_K) \curvearrowright \mathfrak{D}_K^2}(s) &= \frac{1}{(1 - q^{1-2s})(1 - q^{-s})}, \\ \zeta_{U_3(\mathfrak{O}_K) \curvearrowright \mathfrak{D}_K^3}(s) &= \frac{1 - q^{1-4s}}{(1 - q^{2-4s})(1 - q^{1-3s})(1 - q^{1-2s})(1 - q^{-s})}, \\ \zeta_{U_4(\mathfrak{O}_K) \curvearrowright \mathfrak{D}_K^4}(s) &= F_{U_4}(q, q^{-s}) / ((1 - q^{4-8s})(1 - q^{3-7s})(1 - q^{2-6s})(1 - q^{2-5s}) \\ &\quad \times (1 - q^{2-4s})(1 - q^{1-4s})(1 - q^{1-2s})(1 - q^{1-3s})(1 - q^{-s})), \\ \zeta_{U_5(\mathfrak{O}_K) \curvearrowright \mathfrak{D}_K^5}(s) &= F_{U_5}(q, q^{-s}) / ((1 - q^{6-13s})(1 - q^{6-12s})(1 - q^{4-11s}) \\ &\quad \times (1 - q^{4-10s})(1 - q^{3-10s})(1 - q^{4-9s})(1 - q^{3-9s})(1 - q^{4-8s}) \\ &\quad \times (1 - q^{3-8s})(1 - q^{2-8s})(1 - q^{3-7s})(1 - q^{2-7s})(1 - q^{2-6s}) \\ &\quad \times (1 - q^{2-5s})(1 - q^{1-5s})(1 - q^{2-4s})(1 - q^{1-4s})(1 - q^{1-2s}) \\ &\quad \times (1 - q^{-s})),\end{aligned}$$

where

$$\begin{aligned}F_{U_4} &= -X^{10}Y^{30} + X^9Y^{26} + X^9Y^{25} + X^9Y^{24} - X^9Y^{23} + 2X^8Y^{23} - X^8Y^{22} + 2X^7Y^{22} \\ &\quad - 2X^7Y^{21} - 2X^7Y^{20} + X^6Y^{21} - 2X^7Y^{19} + X^6Y^{20} - X^6Y^{18} - X^6Y^{17} - X^5Y^{18} \\ &\quad - X^5Y^{17} + 2X^6Y^{15} - X^5Y^{16} + X^5Y^{14} - 2X^4Y^{15} + X^5Y^{13} + X^5Y^{12} + X^4Y^{13} \\ &\quad + X^4Y^{12} - X^4Y^{10} + 2X^3Y^{11} - X^4Y^9 + 2X^3Y^{10} + 2X^3Y^9 - 2X^3Y^8 + X^2Y^8 \\ &\quad - 2X^2Y^7 + XY^7 - XY^6 - XY^5 - XY^4 + 1\end{aligned}$$

and $F_{U_5} = 1 + \dots + X^{43}Y^{124}$ is given in Appendix A. These formulae for $n \leq 5$ satisfy the functional equation

$$\zeta_{U_n(\mathfrak{O}_K) \curvearrowright \mathfrak{D}_K^n}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^n q^{\binom{n}{2} - \binom{n+1}{2}s} \cdot \zeta_{U_n(\mathfrak{O}_K) \curvearrowright \mathfrak{D}_K^n}(s).$$

Despite the increasing complexity of the formulae in Theorem 9.5, we note that the “reduced submodule zeta function” of $U_n(\mathbf{Z})$ acting on \mathbf{Z}^n (defined and computed using a straightforward variation of [20]) is given by the simple formula $1/((1 - Y)(1 - Y^2) \dots (1 - Y^n))$ for all $n \geq 1$.

Remark 9.6.

- (i) Let \mathfrak{g} be an n -dimensional nilpotent Lie k -algebra. By Engel’s theorem, after choosing a suitable basis, we may regard $\text{ad}(\mathfrak{g})$ as a subset of the enveloping associative algebra $k[U_n(k)]$ of $U_n(k)$ within $M_n(k)$. Suppose that $n > 1$. It is easy to see that the minimal number of generators of $k[U_n(k)]$ as a unital, associative k -algebra is $n - 1$ (use, for instance, [23, p. 263]). Let \mathfrak{z} denote the centre of \mathfrak{g} . Then, as a Lie algebra, $\text{ad}(\mathfrak{g}) \approx \mathfrak{g}/\mathfrak{z}$ is generated by $\dim_k(\mathfrak{g}/([\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}))$ many elements. Hence, if \mathfrak{g} has class ≥ 3 , then $\text{ad}(\mathfrak{g})$ is generated by fewer than $n - 1$ elements. If, on the other hand, \mathfrak{g} has class 2, then $n \geq 3$ and $\text{ad}(\mathfrak{g})$ is an abelian Lie algebra while $k[U_n(k)]$ is non-commutative. We conclude that $\text{ad}(\mathfrak{g})$ never generates all of $k[U_n(k)]$ for $n > 1$.

- (ii) We may sharpen the final statement of (i) as follows. Let $n > 1$. By [38, Prop. 6.1], the abscissa of convergence of $\zeta_{U_n(\mathfrak{o}) \curvearrowright \mathfrak{o}^n}(s)$ is 1. It follows that there does *not* exist a nilpotent Lie \mathfrak{o} -algebra \mathfrak{g} which is finitely generated as an \mathfrak{o} -module such that $\zeta_{U_n(\mathfrak{o}_v) \curvearrowright \mathfrak{o}_v^n}(s) = \zeta_{\mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\triangleleft}(s)$ for almost all $v \in \mathcal{V}_k$. Indeed, it is easy to see that for every finite $S \subset \mathcal{V}_k$, the abscissa of convergence of $\prod_{v \in \mathcal{V}_k \setminus S} \zeta_{\mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\triangleleft}(s)$ is at least $d := \dim_k(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \otimes_{\mathfrak{o}} k)$ (cf. [26, Prop. 1]) and we may clearly assume $d > 1$.

For another illustration of the generally wild effect of direct products of algebraic structures on associated zeta functions, we now consider generic local submodule zeta functions associated with products $U_{n_1} \times \cdots \times U_{n_r}$, diagonally embedded into $U_{n_1 + \cdots + n_r}$.

Theorem 9.7. *For almost all primes p and all finite extensions K/\mathbf{Q}_p , writing $q = q_K$,*

$$\begin{aligned} \zeta_{U_2^2(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^4}(s) &= (1 - q^{2-3s}) / ((1 - q^{3-3s})(1 - q^{2-2s})^2(1 - q^{1-s})(1 - q^{-s})), \\ \zeta_{U_2^3(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s) &= F_{U_2^3}(q, q^{-s}) / ((1 - q^{8-5s})(1 - q^{5-4s})(1 - q^{4-3s})(1 - q^{3-2s})^3 \\ &\quad \times (1 - q^{2-s})(1 - q^{1-s})(1 - q^{-s})), \\ \zeta_{(U_3 \times U_2)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^5}(s) &= F_{U_3 \times U_2}(q, q^{-s}) / ((1 - q^{6-6s})(1 - q^{4-5s})(1 - q^{3-4s})(1 - q^{3-3s}) \\ &\quad \times (1 - q^{2-3s})(1 - q^{2-2s})^2(1 - q^{1-s})(1 - q^{-s})), \\ \zeta_{U_3^2(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s) &= F_{U_3^2}(q, q^{-s}) / ((1 - q^{9-9s})(1 - q^{8-8s})(1 - q^{6-7s})(1 - q^{5-7s}) \\ &\quad \times (1 - q^{6-6s})(1 - q^{4-6s})(1 - q^{4-5s})(1 - q^{3-5s})(1 - q^{3-4s}) \\ &\quad \times (1 - q^{3-3s})(1 - q^{2-3s})(1 - q^{2-2s})^2(1 - q^{1-s})(1 - q^{-s})), \end{aligned}$$

where

$$\begin{aligned} F_{U_2^3} &= -X^{14}Y^{12} + 3X^{11}Y^9 - X^{11}Y^8 - 2X^{10}Y^9 + 2X^{10}Y^8 - X^8Y^7 + 2X^7Y^7 \\ &\quad - 2X^7Y^5 + X^6Y^5 - 2X^4Y^4 + 2X^4Y^3 + X^3Y^4 - 3X^3Y^3 + 1, \\ F_{U_3 \times U_2} &= X^{13}Y^{18} - X^{11}Y^{15} - 2X^{11}Y^{14} + X^{11}Y^{13} + X^{10}Y^{14} - 2X^{10}Y^{13} + X^9Y^{12} \\ &\quad - 2X^8Y^{12} + 3X^8Y^{11} - 2X^7Y^{11} + X^8Y^9 + X^7Y^{10} + X^6Y^8 + X^5Y^9 \\ &\quad - 2X^6Y^7 + 3X^5Y^7 - 2X^5Y^6 + X^4Y^6 - 2X^3Y^5 + X^3Y^4 + X^2Y^5 \\ &\quad - 2X^2Y^4 - X^2Y^3 + 1, \end{aligned}$$

and $F_{U_3^2} = -X^{43}Y^{57} + \cdots + 1$ is given in Appendix A.

These generic local zeta functions satisfy the following functional equations:

$$\begin{aligned} \zeta_{U_2^2(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^4}(s) \Big|_{q \rightarrow q^{-1}} &= q^{6-6s} \cdot \zeta_{U_2^2(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^4}(s), \\ \zeta_{U_2^3(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s) \Big|_{q \rightarrow q^{-1}} &= q^{15-9s} \cdot \zeta_{U_2^3(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s), \\ \zeta_{(U_3 \times U_2)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^5}(s) \Big|_{q \rightarrow q^{-1}} &= -q^{10-9s} \cdot \zeta_{(U_3 \times U_2)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^5}(s), \\ \zeta_{U_3^2(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s) \Big|_{q \rightarrow q^{-1}} &= q^{15-12s} \cdot \zeta_{U_3^2(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s). \end{aligned}$$

Further examples of the above form are included with **Zeta**; here, we only record the following functional equations.

Theorem 9.8. *For almost all primes p and all finite extensions K/\mathbf{Q}_p , writing $q = q_K$,*

$$\begin{aligned} \zeta_{(U_5 \times U_1)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s) \Big|_{q \rightarrow q^{-1}} &= q^{15-16s} \cdot \zeta_{(U_5 \times U_1)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s), \\ \zeta_{(U_3 \times U_2 \times U_1)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s) \Big|_{q \rightarrow q^{-1}} &= q^{15-10s} \cdot \zeta_{(U_3 \times U_2 \times U_1)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s), \\ \zeta_{(U_4 \times U_2)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s) \Big|_{q \rightarrow q^{-1}} &= q^{15-13s} \cdot \zeta_{(U_4 \times U_2)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^6}(s), \\ \zeta_{(U_3 \times U_2^2)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^7}(s) \Big|_{q \rightarrow q^{-1}} &= -q^{21-12s} \cdot \zeta_{(U_3 \times U_2^2)(\mathfrak{D}_K) \curvearrowright \mathfrak{D}_K^7}(s). \end{aligned}$$

We note that the Uniformity Problem has a positive solution for each of the four families of local zeta functions in Theorem 9.8. We further note that the functional equations recorded in Theorems 9.5, 9.7, and 9.8 are explained by [47, Thm 5.5].

10. Applications III: graded subobject zeta functions

By [31, §5.1], up to isomorphism, there are exactly 26 non-abelian fundamental graded Lie \mathbf{C} -algebras (see §3.2) of dimension at most six. All of these algebras are defined in terms of integral structure constants which thus provide us with “natural” \mathbf{Q} -forms. It turns out that for each of the resulting 26 graded Lie \mathbf{Q} -algebras, we can use the techniques developed here to compute their associated generic local graded subalgebra and graded ideal zeta functions. We note that for various of these Lie algebras, the associated non-graded subalgebra and ideal zeta functions are unknown.

Examples of graded ideal zeta functions. Table 2 lists the generic local ideal zeta functions associated with the aforementioned 26 graded Lie \mathbf{Q} -algebras. The first column contains the names of the associated \mathbf{C} -algebras as in [31]; here, an algebra called “ md_c_i ” has dimension d and nilpotency class c .

Given a \mathbf{Z} -form \mathfrak{g} of a graded Lie algebra \mathfrak{g} as indicated by an entry in the first column, the rational function $W(X, Y)$ in the corresponding entry of the second column satisfies the following property: for almost all rational primes p and all finite extensions K/\mathbf{Q}_p , $\zeta_{\mathfrak{g} \otimes \mathfrak{D}_K}^{\text{gr } \triangleleft}(s) = W(q_K, q_K^{-s})$. An entry $\pm X^a Y^b$ in the third column of Table 2 indicates that the corresponding $W(X, Y)$ satisfies $W(X^{-1}, Y^{-1}) = \pm X^a Y^b \cdot W(X, Y)$; an entry “ \mathbf{X} ” signifies the absence of such a functional equation.

The algebras $m6_3_2$ and $m6_3_3$ are precisely the graded Lie algebras associated with $L_{(3,2)}$ in [18, Thm 2.32] (also called L_W [50, Thm 3.4] and $L_{6,25}$ [9]) and $\mathfrak{g}_{6,7}$ in [18, Thm 2.45] (called $L_{6,19}(0)$ in [9]), respectively. The non-graded local ideal zeta functions of these algebras do not satisfy functional equations of the above form either. The algebra $m6_4_1$ is the graded Lie algebra associated with $L_{6,21}(0)$ from [9]; to the author’s knowledge, the non-graded local (and topological) subalgebra and ideal zeta functions of this algebra are unknown.

We note that the formulae for m3_2, m4_3, m5_4_1, and m6_5_1 in Table 2 are consistent with and explained by Proposition 3.5.

Examples of graded subalgebra zeta functions. While the methods developed here can be used to compute the generic local graded subalgebra zeta functions of all 26 algebras in Table 2, we chose to only include the smaller ones of these examples in Table 3 (and Appendix B); for a complete list, we refer to *Zeta* [40].

Open questions. Voll [46, Thm A] established local functional equations under “inversion of p ” for generic local subalgebra zeta functions without any further assumptions on the algebra in question. It is reasonable to expect the following question to have a positive answer; the precise form of (10.1) below was suggested to the author by Voll.

Question 10.1. Let $A = A_1 \oplus \cdots \oplus A_r$ be an \mathfrak{o} -form of a possibly non-associative finite-dimensional k -algebra together with a direct sum decomposition into free \mathfrak{o} -submodules. Let $n = \text{rk}_{\mathfrak{o}}(A)$ and $m = \sum_{i=1}^r \binom{\text{rk}_{\mathfrak{o}}(A_i)}{2}$. Is it always the case that

$$\zeta_{A \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\text{gr} \leq}(s) \Big|_{q_v \rightarrow q_v^{-1}} = (-1)^n q_v^{m-n s} \cdot \zeta_{A \otimes_{\mathfrak{o}} \mathfrak{o}_v}(s) \quad (10.1)$$

for almost all $v \in \mathcal{V}_k$?

The following three questions are graded analogues of conjectures due to Voll (cf. [47, §1.3]).

Question 10.2. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_c$ be a finite-dimensional graded Lie k -algebra of class c . Let $d_i = \dim(\mathfrak{g}_i)$ and $d = \dim(\mathfrak{g})$. Let $0 = \mathfrak{z}_0 \subset \cdots \subset \mathfrak{z}_c = \mathfrak{g}$ be the upper central series of \mathfrak{g} and write $e_i = \dim(\mathfrak{g}/\mathfrak{z}_i)$. Let \mathfrak{g} be an \mathfrak{o} -form of \mathfrak{g} as a graded Lie algebra.

- (i) Does $\zeta_{\mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\text{gr} \triangleleft}(s)$ have degree $-(e_0 + \cdots + e_{c-1})$ in q_v^{-s} for almost all $v \in \mathcal{V}_k$?
- (ii) Suppose that there exists $W \in \mathbf{Q}(X, Y)$ such that $\zeta_{\mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\text{gr} \triangleleft}(s) = W(q_v, q_v^{-s})$ for almost all $v \in \mathcal{V}_k$. Does W have degree $-\binom{d_1}{2} + \cdots + \binom{d_c}{2}$ in X ?
- (iii) Suppose that for almost all $v \in \mathcal{V}_k$,

$$\zeta_{\mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\text{gr} \triangleleft}(s) \Big|_{q_v \rightarrow q_v^{-1}} = \varepsilon q_v^{a-b s} \cdot \zeta_{\mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\text{gr} \triangleleft}(s),$$

where $\varepsilon = \pm 1$ and $a, b \in \mathbf{Z}$. Do we have $\varepsilon = (-1)^d$, $a = \binom{d_1}{2} + \cdots + \binom{d_c}{2}$, and $b = e_0 + \cdots + e_{c-1}$?

Finally, the following is closely related to the questions raised in [35, §8.2].

Question 10.3. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_c$ be a graded nilpotent Lie \mathfrak{o} -algebra of class c , where each \mathfrak{g}_i is free and of finite rank as an \mathfrak{o} -module. Do $\zeta_{\mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\text{gr} \leq}(s)$ and $\zeta_{\mathfrak{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v}^{\text{gr} \triangleleft}(s)$ always have a pole of order c at zero for $v \in \mathcal{V}_k$?

As in [35, §8.2], a natural follow-up question would be to interpret or predict the leading coefficients of the zeta functions in Question 10.3 expanded as Laurent series in s ; however, perhaps unexpectedly, the examples in Tables 2–3 show that these leading coefficients are not functions of v and the numbers $\text{rk}_o(\mathfrak{g}_1), \dots, \text{rk}_o(\mathfrak{g}_c)$ alone.

\mathfrak{g}	$W(X, Y)$ s.t. $\zeta_{\mathfrak{g} \otimes \mathfrak{D}_K}^{\text{gr}}(s) = W(q_K, q_K^{-s})$	FEqn
m3_2	$1/((1 - XY)(1 - Y^3)(1 - Y))$	$-XY^5$
m4_2	$1/((1 - X^2Y)(1 - XY)(1 - Y^3)(1 - Y))$	X^3Y^6
m4_3	$1/((1 - XY)(1 - Y^4)(1 - Y^3)(1 - Y))$	XY^9
m5_2_1	$\frac{1 - Y^6}{(1 - XY^3)(1 - X^2Y)(1 - XY)(1 - Y^5)(1 - Y^3)(1 - Y)}$	$-X^4Y^8$
m5_2_2	$1/((1 - X^3Y)(1 - X^2Y)(1 - XY)(1 - Y^3)(1 - Y))$	$-X^6Y^7$
m5_2_3	$1/((1 - X^3Y)(1 - X^2Y)(1 - XY)(1 - Y^5)(1 - Y))$	$-X^6Y^9$
m5_3_1	$\frac{1 - Y^8}{(1 - XY^4)(1 - XY)(1 - Y^5)(1 - Y^4)(1 - Y^3)(1 - Y)}$	$-X^2Y^{10}$
m5_3_2	$1/((1 - X^2Y)(1 - XY)(1 - Y^4)(1 - Y^3)(1 - Y))$	$-X^3Y^{10}$
m5_4_1	$1/((1 - XY)(1 - Y^5)(1 - Y^4)(1 - Y^3)(1 - Y))$	$-XY^{14}$
m6_2_1	$\frac{XY^8 + XY^5 + Y^5 + XY^3 + Y^3 + 1}{(1 - X^2Y^5)(1 - X^2Y^3)(1 - X^2Y)(1 - XY)(1 - Y^6)(1 - Y)}$	X^6Y^9
m6_2_2	$\frac{1 - Y^6}{(1 - XY^3)(1 - X^3Y)(1 - X^2Y)(1 - XY)(1 - Y^5)(1 - Y^3)(1 - Y)}$	X^7Y^9
m6_2_3	$\frac{Y^4 + Y^3 + Y^2 + Y + 1}{(1 - XY^5)(1 - X^3Y)(1 - X^2Y)(1 - XY)(1 - Y^3)^2}$	X^7Y^{10}
m6_2_4	$\frac{1 - XY^8}{(1 - XY^6)(1 - XY^5)(1 - X^3Y)(1 - X^2Y)(1 - XY)(1 - Y^3)(1 - Y)}$	X^7Y^{10}
m6_2_5	$\frac{1}{(1 - X^4Y)(1 - X^3Y)(1 - X^2Y)(1 - XY)(1 - Y^3)(1 - Y)}$	$X^{10}Y^8$
m6_2_6	$\frac{1}{(1 - X^4Y)(1 - X^3Y)(1 - X^2Y)(1 - XY)(1 - Y^5)(1 - Y)}$	$X^{10}Y^{10}$
m6_3_1	$\frac{1 - Y^8}{(1 - XY^4)(1 - X^2Y)(1 - XY)(1 - Y^5)(1 - Y^4)(1 - Y^3)(1 - Y)}$	X^4Y^{11}
m6_3_2	$\frac{Y^8 + Y^7 + 2Y^6 + 2Y^5 + 2Y^4 + 2Y^3 + Y^2 + Y + 1}{(1 - XY^3)(1 - X^2Y)(1 - XY)(1 - Y^6)(1 - Y^5)(1 - Y^4)}$	\times
m6_3_3	same as for m6_3_2	\times
m6_3_4	$1/((1 - XY^3)(1 - X^2Y)(1 - XY)(1 - Y^5)(1 - Y^3)(1 - Y))$	X^4Y^{14}
m6_3_5	same as for m6_3_4	X^4Y^{14}
m6_3_6	$1/((1 - X^3Y)(1 - X^2Y)(1 - XY)(1 - Y^4)(1 - Y^3)(1 - Y))$	X^6Y^{11}
m6_4_1	$\frac{Y^3 - Y + 1}{(1 - XY^4)(1 - XY)(1 - Y^6)(1 - Y^5)(1 - Y)^2}$	\times
m6_4_2	$\frac{1 - Y^8}{(1 - XY^4)(1 - XY)(1 - Y^6)(1 - Y^5)(1 - Y^4)(1 - Y^3)(1 - Y)}$	X^2Y^{16}
m6_4_3	$1/((1 - X^2Y)(1 - XY)(1 - Y^5)(1 - Y^4)(1 - Y^3)(1 - Y))$	X^3Y^{15}
m6_5_1	$1/((1 - XY)(1 - Y^6)(1 - Y^5)(1 - Y^4)(1 - Y^3)(1 - Y))$	XY^{20}
m6_5_2	same as for m6_5_1	XY^{20}

Table 2: Examples of generic local graded ideal zeta functions

A. Large numerators of local subobject zeta functions

$$F_{U_5} = X^{43}Y^{124} + X^{42}Y^{121} - X^{42}Y^{120} - X^{42}Y^{119} - 2X^{42}Y^{118} + 2X^{41}Y^{118} - 3X^{41}Y^{117} \\ + X^{42}Y^{115} - 2X^{41}Y^{116} + X^{42}Y^{114} - 3X^{41}Y^{115} - 2X^{40}Y^{116} - X^{42}Y^{113} - X^{41}Y^{114}$$

\mathfrak{g}	$W(X, Y)$ s.t. $\zeta_{\mathfrak{g} \otimes \mathcal{D}_K}^{\text{gr} \leq}(s) = W(q_K, q_K^{-s})$	FEqn
m3_2	$\frac{1-XY^3}{(1-XY^2)(1-XY)(1-Y^2)(1-Y)}$	$-XY^3$
m4_2	$\frac{1-XY^3}{(1-X^2Y)(1-XY^2)(1-XY)(1-Y^2)(1-Y)}$	X^3Y^4
m4_3	$\frac{X^2Y^9+X^2Y^7+X^2Y^6-XY^6-2XY^5-2XY^4-XY^3+Y^3+Y^2+1}{(1-XY^3)(1-XY^2)(1-XY)(1-Y^4)(1-Y^2)(1-Y)}$	XY^4
m5_2_1	$\frac{-X^2Y^5-X^2Y^3-XY^3+XY^2+Y^2+1}{(1-X^2Y)(1-XY^2)(1-X^2Y^2)(1-XY)(1-Y^3)(1-Y)}$	$-X^4Y^5$
m5_2_2	$\frac{1-XY^3}{(1-XY^2)(1-X^3Y)(1-X^2Y)(1-XY)(1-Y^2)(1-Y)}$	$-X^6Y^5$
m5_2_3	$\frac{-X^4Y^7-X^3Y^6-X^3Y^4-X^2Y^5+X^3Y^3-XY^4+X^2Y^2+XY^3+XY+1}{(1-X^3Y^3)(1-X^2Y^3)(1-X^3Y)(1-X^2Y)(1-Y^3)(1-Y)}$	$-X^6Y^5$
m5_3_1	W_{531} (B.1)	$-X^2Y^5$
m5_3_2	$\frac{X^2Y^9+X^2Y^7+X^2Y^6-XY^6-2XY^5-2XY^4-XY^3+Y^3+Y^2+1}{(1-XY^3)(1-XY^2)(1-X^2Y)(1-XY)(1-Y^4)(1-Y^2)(1-Y)}$	$-X^3Y^5$
m5_4_1	W_{541} (B.2)	$-XY^5$
m6_2_1	W_{621} (B.3)	X^6Y^6
m6_2_2	$\frac{-X^2Y^5-X^2Y^3-XY^3+XY^2+Y^2+1}{(1-X^2Y^2)(1-XY^2)(1-X^3Y)(1-X^2Y)(1-XY)(1-Y^3)(1-Y)}$	X^7Y^6
m6_2_3	W_{623} (B.4)	X^7Y^6
m6_2_5	$\frac{1-XY^3}{(1-XY^2)(1-X^4Y)(1-X^3Y)(1-X^2Y)(1-XY)(1-Y^2)(1-Y)}$	$X^{10}Y^6$
m6_2_6	$\frac{-X^4Y^7-X^3Y^6-X^3Y^4-X^2Y^5+X^3Y^3-XY^4+X^2Y^2+XY^3+XY+1}{(1-X^3Y^3)(1-X^2Y^3)(1-X^4Y)(1-X^3Y)(1-X^2Y)(1-Y^3)(1-Y)}$	$X^{10}Y^6$
m6_3_1	W_{631} (B.5)	X^4Y^6
m6_3_2	W_{632} (B.6)	X^4Y^6
m6_3_3	W_{633} (B.7)	X^4Y^6
m6_3_6	$\frac{X^2Y^9+X^2Y^7+X^2Y^6-XY^6-2XY^5-2XY^4-XY^3+Y^3+Y^2+1}{(1-XY^3)(1-XY^2)(1-X^3Y)(1-X^2Y)(1-XY)(1-Y^4)(1-Y^2)(1-Y)}$	X^6Y^6
m6_4_3	W_{643} (B.8)	X^3Y^6

Table 3: Examples of generic local graded subalgebra zeta functions

$$\begin{aligned}
& + 2X^{40}Y^{115} + 4X^{41}Y^{113} - 2X^{40}Y^{114} - X^{39}Y^{115} - 2X^{40}Y^{113} - 2X^{39}Y^{114} + X^{41}Y^{111} \\
& + 6X^{40}Y^{112} - 3X^{39}Y^{113} - X^{41}Y^{110} + X^{40}Y^{111} - 4X^{39}Y^{112} + 5X^{40}Y^{110} + 6X^{39}Y^{111} \\
& + X^{38}Y^{112} + 3X^{39}Y^{110} - 6X^{38}Y^{111} - X^{40}Y^{108} + 8X^{39}Y^{109} + 2X^{38}Y^{110} - 2X^{40}Y^{107} \\
& + 4X^{39}Y^{108} + 5X^{38}Y^{109} - 3X^{37}Y^{110} + X^{39}Y^{107} + 9X^{38}Y^{108} - 4X^{39}Y^{106} + 8X^{38}Y^{107} \\
& + X^{37}Y^{108} - 4X^{39}Y^{105} + 3X^{38}Y^{106} + 6X^{37}Y^{107} - 2X^{36}Y^{108} - X^{39}Y^{104} - 5X^{38}Y^{105} \\
& + 13X^{37}Y^{106} + 2X^{36}Y^{107} - 8X^{38}Y^{104} + 9X^{37}Y^{105} + 3X^{36}Y^{106} - X^{35}Y^{107} + 2X^{39}Y^{102} \\
& - 7X^{38}Y^{103} - 3X^{37}Y^{104} + 8X^{36}Y^{105} + X^{35}Y^{106} - 6X^{38}Y^{102} - 15X^{37}Y^{103} + 5X^{36}Y^{104} \\
& - X^{35}Y^{105} + 4X^{38}Y^{101} - 15X^{37}Y^{102} + 6X^{36}Y^{103} + 12X^{35}Y^{104} - X^{38}Y^{100} - 16X^{37}Y^{101} \\
& - 10X^{36}Y^{102} + 7X^{35}Y^{103} - 2X^{34}Y^{104} + 3X^{38}Y^{99} + 4X^{37}Y^{100} - 22X^{36}Y^{101} + 8X^{35}Y^{102} \\
& + 5X^{34}Y^{103} + X^{38}Y^{98} - 28X^{36}Y^{100} - 8X^{35}Y^{101} + 8X^{37}Y^{98} - 8X^{36}Y^{99} - 19X^{35}Y^{100} \\
& + 13X^{34}Y^{101} + 2X^{33}Y^{102} + X^{37}Y^{97} - 2X^{36}Y^{98} - 30X^{35}Y^{99} + X^{34}Y^{100} + X^{33}Y^{101} \\
& - X^{37}Y^{96} + 17X^{36}Y^{97} - 19X^{35}Y^{98} - 16X^{34}Y^{99} + 6X^{33}Y^{100} + 17X^{36}Y^{96} - 6X^{35}Y^{97} \\
& - 34X^{34}Y^{98} + 2X^{33}Y^{99} + X^{32}Y^{100} - X^{37}Y^{94} + 7X^{36}Y^{95} + 16X^{35}Y^{96} - 32X^{34}Y^{97} \\
& + 2X^{32}Y^{99} - X^{36}Y^{94} + 32X^{35}Y^{95} - 15X^{34}Y^{96} - 21X^{33}Y^{97} + 2X^{32}Y^{98} - 5X^{36}Y^{93} \\
& + 16X^{35}Y^{94} + 4X^{34}Y^{95} - 32X^{33}Y^{96} - 2X^{32}Y^{97} - X^{36}Y^{92} + 17X^{35}Y^{93} + 56X^{34}Y^{94} \\
& - 22X^{33}Y^{95} - 13X^{32}Y^{96} + 3X^{31}Y^{97} - X^{36}Y^{91} - 5X^{35}Y^{92} + 30X^{34}Y^{93} - 13X^{33}Y^{94} \\
& - 22X^{32}Y^{95} - 7X^{35}Y^{91} + 37X^{34}Y^{92} + 59X^{33}Y^{93} - 25X^{32}Y^{94} - 5X^{31}Y^{95} - 3X^{35}Y^{90}
\end{aligned}$$

$$\begin{aligned}
& -5X^{34}Y^{91} + 41X^{33}Y^{92} - 29X^{32}Y^{93} - 17X^{31}Y^{94} - X^{35}Y^{89} - 17X^{34}Y^{90} + 69X^{33}Y^{91} \\
& + 41X^{32}Y^{92} - 22X^{31}Y^{93} + X^{30}Y^{94} - 11X^{34}Y^{89} + 18X^{33}Y^{90} + 48X^{32}Y^{91} - 26X^{31}Y^{92} \\
& - 5X^{30}Y^{93} - 12X^{34}Y^{88} - 26X^{33}Y^{89} + 89X^{32}Y^{90} + 13X^{31}Y^{91} - 10X^{30}Y^{92} - X^{29}Y^{93} \\
& + 2X^{34}Y^{87} - 26X^{33}Y^{88} + 52X^{32}Y^{89} + 33X^{31}Y^{90} - 23X^{30}Y^{91} - 3X^{29}Y^{92} + 3X^{34}Y^{86} \\
& - 34X^{33}Y^{87} - 24X^{32}Y^{88} + 82X^{31}Y^{89} - 4X^{29}Y^{91} + X^{34}Y^{85} - 10X^{33}Y^{86} - 38X^{32}Y^{87} \\
& + 88X^{31}Y^{88} + 24X^{30}Y^{89} - 11X^{29}Y^{90} - 3X^{33}Y^{85} - 69X^{32}Y^{86} + 4X^{31}Y^{87} + 66X^{30}Y^{88} \\
& - 6X^{29}Y^{89} - 2X^{28}Y^{90} + 7X^{33}Y^{84} - 30X^{32}Y^{85} - 31X^{31}Y^{86} + 101X^{30}Y^{87} - 5X^{28}Y^{89} \\
& + 4X^{33}Y^{83} - 11X^{32}Y^{84} - 103X^{31}Y^{85} + 28X^{30}Y^{86} + 37X^{29}Y^{87} - 4X^{28}Y^{88} + 10X^{32}Y^{83} \\
& - 77X^{31}Y^{84} - 5X^{30}Y^{85} + 91X^{29}Y^{86} - 2X^{28}Y^{87} + X^{33}Y^{81} + 9X^{32}Y^{82} - 40X^{31}Y^{83} \\
& - 99X^{30}Y^{84} + 53X^{29}Y^{85} + 21X^{28}Y^{86} - 2X^{27}Y^{87} + 4X^{32}Y^{81} + 9X^{31}Y^{82} - 115X^{30}Y^{83} \\
& + 20X^{29}Y^{84} + 53X^{28}Y^{85} - 6X^{27}Y^{86} + 4X^{32}Y^{80} + 32X^{31}Y^{81} - 78X^{30}Y^{82} - 80X^{29}Y^{83} \\
& + 56X^{28}Y^{84} + 5X^{27}Y^{85} - 2X^{32}Y^{79} + 22X^{31}Y^{80} - 18X^{30}Y^{81} - 148X^{29}Y^{82} + 44X^{28}Y^{83} \\
& + 30X^{27}Y^{84} - X^{26}Y^{85} + 11X^{31}Y^{79} + 42X^{30}Y^{80} - 114X^{29}Y^{81} - 25X^{28}Y^{82} + 46X^{27}Y^{83} \\
& + 3X^{31}Y^{78} + 56X^{30}Y^{79} - 64X^{29}Y^{80} - 138X^{28}Y^{81} + 32X^{27}Y^{82} + 6X^{26}Y^{83} - X^{31}Y^{77} \\
& + 36X^{30}Y^{78} + 35X^{29}Y^{79} - 143X^{28}Y^{80} + 10X^{27}Y^{81} + 22X^{26}Y^{82} - 3X^{31}Y^{76} + 14X^{30}Y^{77} \\
& + 89X^{29}Y^{78} - 118X^{28}Y^{79} - 100X^{27}Y^{80} + 29X^{26}Y^{81} + 4X^{25}Y^{82} + X^{31}Y^{75} - 3X^{30}Y^{76} \\
& + 76X^{29}Y^{77} - 130X^{27}Y^{79} + 33X^{26}Y^{80} + 10X^{25}Y^{81} - 7X^{30}Y^{75} + 50X^{29}Y^{76} + 107X^{28}Y^{77} \\
& - 152X^{27}Y^{78} - 62X^{26}Y^{79} + 13X^{25}Y^{80} + X^{30}Y^{74} + 6X^{29}Y^{75} + 121X^{28}Y^{76} - 71X^{27}Y^{77} \\
& - 99X^{26}Y^{78} + 23X^{25}Y^{79} + X^{24}Y^{80} - 3X^{30}Y^{73} - 19X^{29}Y^{74} + 100X^{28}Y^{75} + 88X^{27}Y^{76} \\
& - 137X^{26}Y^{77} - 16X^{25}Y^{78} + 8X^{24}Y^{79} - 12X^{29}Y^{73} + 34X^{28}Y^{74} + 145X^{27}Y^{75} - 119X^{26}Y^{76} \\
& - 53X^{25}Y^{77} + 16X^{24}Y^{78} - 7X^{29}Y^{72} - 16X^{28}Y^{73} + 167X^{27}Y^{74} + 47X^{26}Y^{75} - 107X^{25}Y^{76} \\
& - 3X^{24}Y^{77} + X^{23}Y^{78} - X^{29}Y^{71} - 28X^{28}Y^{72} + 84X^{27}Y^{73} + 118X^{26}Y^{74} - 132X^{25}Y^{75} \\
& - 29X^{24}Y^{76} + 4X^{23}Y^{77} - 2X^{29}Y^{70} - 27X^{28}Y^{71} - 12X^{27}Y^{72} + 209X^{26}Y^{73} + 6X^{25}Y^{74} \\
& - 55X^{24}Y^{75} + 5X^{23}Y^{76} - 13X^{28}Y^{70} - 54X^{27}Y^{71} + 156X^{26}Y^{72} + 80X^{25}Y^{73} - 117X^{24}Y^{74} \\
& - 6X^{23}Y^{75} + X^{22}Y^{76} - X^{28}Y^{69} - 57X^{27}Y^{70} + 38X^{26}Y^{71} + 213X^{25}Y^{72} - 39X^{24}Y^{73} \\
& - 19X^{23}Y^{74} + X^{22}Y^{75} + 2X^{28}Y^{68} - 33X^{27}Y^{69} - 51X^{26}Y^{70} + 208X^{25}Y^{71} + 28X^{24}Y^{72} \\
& - 77X^{23}Y^{73} - 4X^{22}Y^{74} - 13X^{27}Y^{68} - 114X^{26}Y^{69} + 107X^{25}Y^{70} + 164X^{24}Y^{71} - 41X^{23}Y^{72} \\
& - 2X^{22}Y^{73} - 80X^{26}Y^{68} - 11X^{25}Y^{69} + 221X^{24}Y^{70} - 5X^{23}Y^{71} - 35X^{22}Y^{72} - X^{21}Y^{73} \\
& + 6X^{27}Y^{66} - 35X^{26}Y^{67} - 132X^{25}Y^{68} + 154X^{24}Y^{69} + 87X^{23}Y^{70} - 36X^{22}Y^{71} - X^{21}Y^{72} \\
& + 3X^{27}Y^{65} - 131X^{25}Y^{67} + 44X^{24}Y^{68} + 200X^{23}Y^{69} - 22X^{22}Y^{70} - 14X^{21}Y^{71} + 10X^{26}Y^{65} \\
& - 97X^{25}Y^{66} - 124X^{24}Y^{67} + 176X^{23}Y^{68} + 33X^{22}Y^{69} - 21X^{21}Y^{70} + 11X^{26}Y^{64} - 18X^{25}Y^{65} \\
& - 177X^{24}Y^{66} + 109X^{23}Y^{67} + 135X^{22}Y^{68} - 15X^{21}Y^{69} - X^{20}Y^{70} + 7X^{26}Y^{63} + 21X^{25}Y^{64} \\
& - 162X^{24}Y^{65} - 73X^{23}Y^{66} + 144X^{22}Y^{67} - 3X^{21}Y^{68} - 8X^{20}Y^{69} + 2X^{26}Y^{62} + 35X^{25}Y^{63} \\
& - 62X^{24}Y^{64} - 189X^{23}Y^{65} + 145X^{22}Y^{66} + 71X^{21}Y^{67} - 8X^{20}Y^{68} + X^{26}Y^{61} + 16X^{25}Y^{62} \\
& - X^{24}Y^{63} - 216X^{23}Y^{64} - 7X^{22}Y^{65} + 101X^{21}Y^{66} - 9X^{20}Y^{67} - 2X^{19}Y^{68} + 5X^{25}Y^{61} \\
& + 63X^{24}Y^{62} - 131X^{23}Y^{63} - 155X^{22}Y^{64} + 132X^{21}Y^{65} + 36X^{20}Y^{66} + 5X^{25}Y^{60} + 51X^{24}Y^{61} \\
& - 32X^{23}Y^{62} - 241X^{22}Y^{63} + 48X^{21}Y^{64} + 53X^{20}Y^{65} - 8X^{19}Y^{66} - 2X^{25}Y^{59} + 21X^{24}Y^{60} \\
& + 76X^{23}Y^{61} - 205X^{22}Y^{62} - 92X^{21}Y^{63} + 86X^{20}Y^{64} + 10X^{19}Y^{65} + 10X^{24}Y^{59} + 86X^{23}Y^{60} \\
& - 92X^{22}Y^{61} - 205X^{21}Y^{62} + 76X^{20}Y^{63} + 21X^{19}Y^{64} - 2X^{18}Y^{65} - 8X^{24}Y^{58} + 53X^{23}Y^{59} \\
& + 48X^{22}Y^{60} - 241X^{21}Y^{61} - 32X^{20}Y^{62} + 51X^{19}Y^{63} + 5X^{18}Y^{64} + 36X^{23}Y^{58} + 132X^{22}Y^{59} \\
& - 155X^{21}Y^{60} - 131X^{20}Y^{61} + 63X^{19}Y^{62} + 5X^{18}Y^{63} - 2X^{24}Y^{56} - 9X^{23}Y^{57} + 101X^{22}Y^{58} \\
& - 7X^{21}Y^{59} - 216X^{20}Y^{60} - X^{19}Y^{61} + 16X^{18}Y^{62} + X^{17}Y^{63} - 8X^{23}Y^{56} + 71X^{22}Y^{57}
\end{aligned}$$

$$\begin{aligned}
& + 145X^{21}Y^{58} - 189X^{20}Y^{59} - 62X^{19}Y^{60} + 35X^{18}Y^{61} + 2X^{17}Y^{62} - 8X^{23}Y^{55} - 3X^{22}Y^{56} \\
& + 144X^{21}Y^{57} - 73X^{20}Y^{58} - 162X^{19}Y^{59} + 21X^{18}Y^{60} + 7X^{17}Y^{61} - X^{23}Y^{54} - 15X^{22}Y^{55} \\
& + 135X^{21}Y^{56} + 109X^{20}Y^{57} - 177X^{19}Y^{58} - 18X^{18}Y^{59} + 11X^{17}Y^{60} - 21X^{22}Y^{54} + 33X^{21}Y^{55} \\
& + 176X^{20}Y^{56} - 124X^{19}Y^{57} - 97X^{18}Y^{58} + 10X^{17}Y^{59} - 14X^{22}Y^{53} - 22X^{21}Y^{54} + 200X^{20}Y^{55} \\
& + 44X^{19}Y^{56} - 131X^{18}Y^{57} + 3X^{16}Y^{59} - X^{22}Y^{52} - 36X^{21}Y^{53} + 87X^{20}Y^{54} + 154X^{19}Y^{55} \\
& - 132X^{18}Y^{56} - 35X^{17}Y^{57} + 6X^{16}Y^{58} - X^{22}Y^{51} - 35X^{21}Y^{52} - 5X^{20}Y^{53} + 221X^{19}Y^{54} \\
& - 11X^{18}Y^{55} - 80X^{17}Y^{56} - 2X^{21}Y^{51} - 41X^{20}Y^{52} + 164X^{19}Y^{53} + 107X^{18}Y^{54} - 114X^{17}Y^{55} \\
& - 13X^{16}Y^{56} - 4X^{21}Y^{50} - 77X^{20}Y^{51} + 28X^{19}Y^{52} + 208X^{18}Y^{53} - 51X^{17}Y^{54} - 33X^{16}Y^{55} \\
& + 2X^{15}Y^{56} + X^{21}Y^{49} - 19X^{20}Y^{50} - 39X^{19}Y^{51} + 213X^{18}Y^{52} + 38X^{17}Y^{53} - 57X^{16}Y^{54} \\
& - X^{15}Y^{55} + X^{21}Y^{48} - 6X^{20}Y^{49} - 117X^{19}Y^{50} + 80X^{18}Y^{51} + 156X^{17}Y^{52} - 54X^{16}Y^{53} \\
& - 13X^{15}Y^{54} + 5X^{20}Y^{48} - 55X^{19}Y^{49} + 6X^{18}Y^{50} + 209X^{17}Y^{51} - 12X^{16}Y^{52} - 27X^{15}Y^{53} \\
& - 2X^{14}Y^{54} + 4X^{20}Y^{47} - 29X^{19}Y^{48} - 132X^{18}Y^{49} + 118X^{17}Y^{50} + 84X^{16}Y^{51} - 28X^{15}Y^{52} \\
& - X^{14}Y^{53} + X^{20}Y^{46} - 3X^{19}Y^{47} - 107X^{18}Y^{48} + 47X^{17}Y^{49} + 167X^{16}Y^{50} - 16X^{15}Y^{51} \\
& - 7X^{14}Y^{52} + 16X^{19}Y^{46} - 53X^{18}Y^{47} - 119X^{17}Y^{48} + 145X^{16}Y^{49} + 34X^{15}Y^{50} - 12X^{14}Y^{51} \\
& + 8X^{19}Y^{45} - 16X^{18}Y^{46} - 137X^{17}Y^{47} + 88X^{16}Y^{48} + 100X^{15}Y^{49} - 19X^{14}Y^{50} - 3X^{13}Y^{51} \\
& + X^{19}Y^{44} + 23X^{18}Y^{45} - 99X^{17}Y^{46} - 71X^{16}Y^{47} + 121X^{15}Y^{48} + 6X^{14}Y^{49} + X^{13}Y^{50} \\
& + 13X^{18}Y^{44} - 62X^{17}Y^{45} - 152X^{16}Y^{46} + 107X^{15}Y^{47} + 50X^{14}Y^{48} - 7X^{13}Y^{49} + 10X^{18}Y^{43} \\
& + 33X^{17}Y^{44} - 130X^{16}Y^{45} + 76X^{14}Y^{47} - 3X^{13}Y^{48} + X^{12}Y^{49} + 4X^{18}Y^{42} + 29X^{17}Y^{43} \\
& - 100X^{16}Y^{44} - 118X^{15}Y^{45} + 89X^{14}Y^{46} + 14X^{13}Y^{47} - 3X^{12}Y^{48} + 22X^{17}Y^{42} + 10X^{16}Y^{43} \\
& - 143X^{15}Y^{44} + 35X^{14}Y^{45} + 36X^{13}Y^{46} - X^{12}Y^{47} + 6X^{17}Y^{41} + 32X^{16}Y^{42} - 138X^{15}Y^{43} \\
& - 64X^{14}Y^{44} + 56X^{13}Y^{45} + 3X^{12}Y^{46} + 46X^{16}Y^{41} - 25X^{15}Y^{42} - 114X^{14}Y^{43} + 42X^{13}Y^{44} \\
& + 11X^{12}Y^{45} - X^{17}Y^{39} + 30X^{16}Y^{40} + 44X^{15}Y^{41} - 148X^{14}Y^{42} - 18X^{13}Y^{43} + 22X^{12}Y^{44} \\
& - 2X^{11}Y^{45} + 5X^{16}Y^{39} + 56X^{15}Y^{40} - 80X^{14}Y^{41} - 78X^{13}Y^{42} + 32X^{12}Y^{43} + 4X^{11}Y^{44} \\
& - 6X^{16}Y^{38} + 53X^{15}Y^{39} + 20X^{14}Y^{40} - 115X^{13}Y^{41} + 9X^{12}Y^{42} + 4X^{11}Y^{43} - 2X^{16}Y^{37} \\
& + 21X^{15}Y^{38} + 53X^{14}Y^{39} - 99X^{13}Y^{40} - 40X^{12}Y^{41} + 9X^{11}Y^{42} + X^{10}Y^{43} - 2X^{15}Y^{37} \\
& + 91X^{14}Y^{38} - 5X^{13}Y^{39} - 77X^{12}Y^{40} + 10X^{11}Y^{41} - 4X^{15}Y^{36} + 37X^{14}Y^{37} + 28X^{13}Y^{38} \\
& - 103X^{12}Y^{39} - 11X^{11}Y^{40} + 4X^{10}Y^{41} - 5X^{15}Y^{35} + 101X^{13}Y^{37} - 31X^{12}Y^{38} - 30X^{11}Y^{39} \\
& + 7X^{10}Y^{40} - 2X^{15}Y^{34} - 6X^{14}Y^{35} + 66X^{13}Y^{36} + 4X^{12}Y^{37} - 69X^{11}Y^{38} - 3X^{10}Y^{39} \\
& - 11X^{14}Y^{34} + 24X^{13}Y^{35} + 88X^{12}Y^{36} - 38X^{11}Y^{37} - 10X^{10}Y^{38} + X^9Y^{39} - 4X^{14}Y^{33} \\
& + 82X^{12}Y^{35} - 24X^{11}Y^{36} - 34X^{10}Y^{37} + 3X^9Y^{38} - 3X^{14}Y^{32} - 23X^{13}Y^{33} + 33X^{12}Y^{34} \\
& + 52X^{11}Y^{35} - 26X^{10}Y^{36} + 2X^9Y^{37} - X^{14}Y^{31} - 10X^{13}Y^{32} + 13X^{12}Y^{33} + 89X^{11}Y^{34} \\
& - 26X^{10}Y^{35} - 12X^9Y^{36} - 5X^{13}Y^{31} - 26X^{12}Y^{32} + 48X^{11}Y^{33} + 18X^{10}Y^{34} - 11X^9Y^{35} \\
& + X^{13}Y^{30} - 22X^{12}Y^{31} + 41X^{11}Y^{32} + 69X^{10}Y^{33} - 17X^9Y^{34} - X^8Y^{35} - 17X^{12}Y^{30} \\
& - 29X^{11}Y^{31} + 41X^{10}Y^{32} - 5X^9Y^{33} - 3X^8Y^{34} - 5X^{12}Y^{29} - 25X^{11}Y^{30} + 59X^{10}Y^{31} \\
& + 37X^9Y^{32} - 7X^8Y^{33} - 22X^{11}Y^{29} - 13X^{10}Y^{30} + 30X^9Y^{31} - 5X^8Y^{32} - X^7Y^{33} \\
& + 3X^{12}Y^{27} - 13X^{11}Y^{28} - 22X^{10}Y^{29} + 56X^9Y^{30} + 17X^8Y^{31} - X^7Y^{32} - 2X^{11}Y^{27} \\
& - 32X^{10}Y^{28} + 4X^9Y^{29} + 16X^8Y^{30} - 5X^7Y^{31} + 2X^{11}Y^{26} - 21X^{10}Y^{27} - 15X^9Y^{28} \\
& + 32X^8Y^{29} - X^7Y^{30} + 2X^{11}Y^{25} - 32X^9Y^{27} + 16X^8Y^{28} + 7X^7Y^{29} - X^6Y^{30} \\
& + X^{11}Y^{24} + 2X^{10}Y^{25} - 34X^9Y^{26} - 6X^8Y^{27} + 17X^7Y^{28} + 6X^{10}Y^{24} - 16X^9Y^{25} \\
& - 19X^8Y^{26} + 17X^7Y^{27} - X^6Y^{28} + X^{10}Y^{23} + X^9Y^{24} - 30X^8Y^{25} - 2X^7Y^{26} \\
& + X^6Y^{27} + 2X^{10}Y^{22} + 13X^9Y^{23} - 19X^8Y^{24} - 8X^7Y^{25} + 8X^6Y^{26} - 8X^8Y^{23} \\
& - 28X^7Y^{24} + X^5Y^{26} + 5X^9Y^{21} + 8X^8Y^{22} - 22X^7Y^{23} + 4X^6Y^{24} + 3X^5Y^{25}
\end{aligned}$$

$$\begin{aligned}
& -2X^9Y^{20} + 7X^8Y^{21} - 10X^7Y^{22} - 16X^6Y^{23} - X^5Y^{24} + 12X^8Y^{20} + 6X^7Y^{21} \\
& - 15X^6Y^{22} + 4X^5Y^{23} - X^8Y^{19} + 5X^7Y^{20} - 15X^6Y^{21} - 6X^5Y^{22} + X^8Y^{18} \\
& + 8X^7Y^{19} - 3X^6Y^{20} - 7X^5Y^{21} + 2X^4Y^{22} - X^8Y^{17} + 3X^7Y^{18} + 9X^6Y^{19} \\
& - 8X^5Y^{20} + 2X^7Y^{17} + 13X^6Y^{18} - 5X^5Y^{19} - X^4Y^{20} - 2X^7Y^{16} + 6X^6Y^{17} \\
& + 3X^5Y^{18} - 4X^4Y^{19} + X^6Y^{16} + 8X^5Y^{17} - 4X^4Y^{18} + 9X^5Y^{16} + X^4Y^{17} - 3X^6Y^{14} \\
& + 5X^5Y^{15} + 4X^4Y^{16} - 2X^3Y^{17} + 2X^5Y^{14} + 8X^4Y^{15} - X^3Y^{16} - 6X^5Y^{13} + 3X^4Y^{14} \\
& + X^5Y^{12} + 6X^4Y^{13} + 5X^3Y^{14} - 4X^4Y^{12} + X^3Y^{13} - X^2Y^{14} - 3X^4Y^{11} + 6X^3Y^{12} + X^2Y^{13} \\
& - 2X^4Y^{10} - 2X^3Y^{11} - X^4Y^9 - 2X^3Y^{10} + 4X^2Y^{11} + 2X^3Y^9 - X^2Y^{10} - XY^{11} - 2X^3Y^8 \\
& - 3X^2Y^9 + XY^{10} - 2X^2Y^8 + XY^9 - 3X^2Y^7 + 2X^2Y^6 - 2XY^6 - XY^5 - XY^4 + XY^3 + 1
\end{aligned}$$

$$\begin{aligned}
F_{\mathbf{Q}[T]/T^4} = & -X^{49}Y^{54} - X^{49}Y^{53} - 2X^{48}Y^{52} - X^{47}Y^{52} - 3X^{47}Y^{51} - 2X^{46}Y^{51} - 3X^{46}Y^{50} \\
& - 4X^{45}Y^{50} + X^{46}Y^{48} - 4X^{45}Y^{49} + X^{45}Y^{48} - 4X^{44}Y^{49} + X^{45}Y^{47} - 2X^{44}Y^{48} \\
& + 3X^{44}Y^{47} - 8X^{43}Y^{48} + X^{44}Y^{46} - X^{43}Y^{47} + 8X^{43}Y^{46} - 9X^{42}Y^{47} + X^{43}Y^{45} \\
& + 3X^{42}Y^{46} + 9X^{42}Y^{45} - 12X^{41}Y^{46} + X^{42}Y^{44} + 10X^{41}Y^{45} + 10X^{41}Y^{44} - 13X^{40}Y^{45} \\
& + 23X^{40}Y^{44} + 7X^{40}Y^{43} - 19X^{39}Y^{44} - 3X^{40}Y^{42} + 35X^{39}Y^{43} + 3X^{39}Y^{42} - 19X^{38}Y^{43} \\
& - 3X^{39}Y^{41} + 54X^{38}Y^{42} - 15X^{38}Y^{41} - 24X^{37}Y^{42} - 6X^{38}Y^{40} + 74X^{37}Y^{41} - 31X^{37}Y^{40} \\
& - 25X^{36}Y^{41} - 5X^{37}Y^{39} + 95X^{36}Y^{40} - 55X^{36}Y^{39} - 30X^{35}Y^{40} - 4X^{36}Y^{38} + 110X^{35}Y^{39} \\
& - X^{36}Y^{37} - 85X^{35}Y^{38} - 28X^{34}Y^{39} + 10X^{35}Y^{37} + 131X^{34}Y^{38} - 3X^{35}Y^{36} - 127X^{34}Y^{37} \\
& - 31X^{33}Y^{38} + 22X^{34}Y^{36} + 143X^{33}Y^{37} - 4X^{34}Y^{35} - 160X^{33}Y^{36} - 29X^{32}Y^{37} + 46X^{33}Y^{35} \\
& + 154X^{32}Y^{36} - 8X^{33}Y^{34} - 204X^{32}Y^{35} - 30X^{31}Y^{36} + 73X^{32}Y^{34} + 159X^{31}Y^{35} - 11X^{32}Y^{33} \\
& - 246X^{31}Y^{34} - 26X^{30}Y^{35} + X^{32}Y^{32} + 113X^{31}Y^{33} + 169X^{30}Y^{34} - 19X^{31}Y^{32} - 290X^{30}Y^{33} \\
& - 27X^{29}Y^{34} + X^{31}Y^{31} + 148X^{30}Y^{32} + 166X^{29}Y^{33} - 26X^{30}Y^{31} - 314X^{29}Y^{32} - 23X^{28}Y^{33} \\
& + 3X^{30}Y^{30} + 193X^{29}Y^{31} + 162X^{28}Y^{32} - 39X^{29}Y^{30} - 344X^{28}Y^{31} - 22X^{27}Y^{32} + 3X^{29}Y^{29} \\
& + 230X^{28}Y^{30} + 153X^{27}Y^{31} - 49X^{28}Y^{29} - 354X^{27}Y^{30} - 17X^{26}Y^{31} + 6X^{28}Y^{28} + 271X^{27}Y^{29} \\
& + 142X^{26}Y^{30} - 68X^{27}Y^{28} - 359X^{26}Y^{29} - 16X^{25}Y^{30} + 6X^{27}Y^{27} + 301X^{26}Y^{28} + 121X^{25}Y^{29} \\
& - 85X^{26}Y^{27} - 344X^{25}Y^{28} - 11X^{24}Y^{29} + 10X^{26}Y^{26} + 332X^{25}Y^{27} + 104X^{24}Y^{28} - 104X^{25}Y^{26} \\
& - 332X^{24}Y^{27} - 10X^{23}Y^{28} + 11X^{25}Y^{25} + 344X^{24}Y^{26} + 85X^{23}Y^{27} - 121X^{24}Y^{25} - 301X^{23}Y^{26} \\
& - 6X^{22}Y^{27} + 16X^{24}Y^{24} + 359X^{23}Y^{25} + 68X^{22}Y^{26} - 142X^{23}Y^{24} - 271X^{22}Y^{25} - 6X^{21}Y^{26} \\
& + 17X^{23}Y^{23} + 354X^{22}Y^{24} + 49X^{21}Y^{25} - 153X^{22}Y^{23} - 230X^{21}Y^{24} - 3X^{20}Y^{25} + 22X^{22}Y^{22} \\
& + 344X^{21}Y^{23} + 39X^{20}Y^{24} - 162X^{21}Y^{22} - 193X^{20}Y^{23} - 3X^{19}Y^{24} + 23X^{21}Y^{21} + 314X^{20}Y^{22} \\
& + 26X^{19}Y^{23} - 166X^{20}Y^{21} - 148X^{19}Y^{22} - X^{18}Y^{23} + 27X^{20}Y^{20} + 290X^{19}Y^{21} + 19X^{18}Y^{22} \\
& - 169X^{19}Y^{20} - 113X^{18}Y^{21} - X^{17}Y^{22} + 26X^{19}Y^{19} + 246X^{18}Y^{20} + 11X^{17}Y^{21} - 159X^{18}Y^{19} \\
& - 73X^{17}Y^{20} + 30X^{18}Y^{18} + 204X^{17}Y^{19} + 8X^{16}Y^{20} - 154X^{17}Y^{18} - 46X^{16}Y^{19} + 29X^{17}Y^{17} \\
& + 160X^{16}Y^{18} + 4X^{15}Y^{19} - 143X^{16}Y^{17} - 22X^{15}Y^{18} + 31X^{16}Y^{16} + 127X^{15}Y^{17} + 3X^{14}Y^{18} \\
& - 131X^{15}Y^{16} - 10X^{14}Y^{17} + 28X^{15}Y^{15} + 85X^{14}Y^{16} + X^{13}Y^{17} - 110X^{14}Y^{15} + 4X^{13}Y^{16} \\
& + 30X^{14}Y^{14} + 55X^{13}Y^{15} - 95X^{13}Y^{14} + 5X^{12}Y^{15} + 25X^{13}Y^{13} + 31X^{12}Y^{14} - 74X^{12}Y^{13} \\
& + 6X^{11}Y^{14} + 24X^{12}Y^{12} + 15X^{11}Y^{13} - 54X^{11}Y^{12} + 3X^{10}Y^{13} + 19X^{11}Y^{11} - 3X^{10}Y^{12} \\
& - 35X^{10}Y^{11} + 3X^9Y^{12} + 19X^{10}Y^{10} - 7X^9Y^{11} - 23X^9Y^{10} + 13X^9Y^9 - 10X^8Y^{10} \\
& - 10X^8Y^9 - X^7Y^{10} + 12X^8Y^8 - 9X^7Y^9 - 3X^7Y^8 - X^6Y^9 + 9X^7Y^7 \\
& - 8X^6Y^8 + X^6Y^7 - X^5Y^8 + 8X^6Y^6 - 3X^5Y^7 + 2X^5Y^6 - X^4Y^7 \\
& + 4X^5Y^5 - X^4Y^6 + 4X^4Y^5 - X^3Y^6 + 4X^4Y^4 + 3X^3Y^4 + 2X^3Y^3
\end{aligned}$$

$$+ 3X^2Y^3 + X^2Y^2 + 2XY^2 + Y + 1$$

$$\begin{aligned}
F_{U_3^2} = & -X^{43}Y^{57} + 4X^{41}Y^{53} - X^{41}Y^{52} - 3X^{40}Y^{53} - 2X^{41}Y^{51} + 4X^{40}Y^{52} + X^{41}Y^{50} + 4X^{40}Y^{51} \\
& - X^{39}Y^{52} - 3X^{40}Y^{50} - 3X^{39}Y^{51} + 5X^{39}Y^{50} + 4X^{38}Y^{51} - 4X^{38}Y^{50} - X^{37}Y^{51} - 2X^{39}Y^{48} \\
& - 3X^{38}Y^{49} + 3X^{37}Y^{50} + X^{39}Y^{47} - X^{38}Y^{48} + 5X^{37}Y^{49} - 3X^{38}Y^{47} - 3X^{37}Y^{48} - X^{36}Y^{49} \\
& + 3X^{38}Y^{46} + 4X^{36}Y^{48} - 6X^{37}Y^{46} - 2X^{36}Y^{47} + X^{35}Y^{48} - X^{37}Y^{45} - X^{36}Y^{46} - 2X^{35}Y^{47} \\
& + 2X^{37}Y^{44} + 4X^{36}Y^{45} - 4X^{35}Y^{46} + 3X^{34}Y^{47} - 3X^{36}Y^{44} - 6X^{35}Y^{45} + X^{34}Y^{46} - X^{36}Y^{43} \\
& + 8X^{35}Y^{44} - 6X^{34}Y^{45} + X^{33}Y^{46} + 6X^{35}Y^{43} - 9X^{34}Y^{44} + 4X^{33}Y^{45} - X^{35}Y^{42} - 4X^{34}Y^{43} \\
& - 6X^{33}Y^{44} - X^{32}Y^{45} - 2X^{35}Y^{41} + 13X^{34}Y^{42} + 4X^{33}Y^{43} - 2X^{32}Y^{44} + 7X^{34}Y^{41} - 11X^{33}Y^{42} \\
& - 6X^{32}Y^{43} + 3X^{31}Y^{44} - 4X^{34}Y^{40} - 2X^{33}Y^{41} + 10X^{32}Y^{42} - 4X^{31}Y^{43} + 10X^{33}Y^{40} + 19X^{32}Y^{41} \\
& - 8X^{31}Y^{42} + X^{30}Y^{43} - 2X^{33}Y^{39} - 16X^{32}Y^{40} - 21X^{31}Y^{41} - 2X^{30}Y^{42} - 7X^{32}Y^{39} + 40X^{31}Y^{40} \\
& + 6X^{30}Y^{41} - X^{29}Y^{42} + 8X^{32}Y^{38} + 19X^{31}Y^{39} - 14X^{30}Y^{40} - 3X^{32}Y^{37} - 30X^{31}Y^{38} - 9X^{30}Y^{39} \\
& - 9X^{29}Y^{40} - 2X^{28}Y^{41} + 2X^{31}Y^{37} + 32X^{30}Y^{38} + 32X^{29}Y^{39} + 2X^{28}Y^{40} + 2X^{30}Y^{37} - 10X^{29}Y^{38} \\
& - 19X^{28}Y^{39} - 13X^{30}Y^{36} - 26X^{29}Y^{37} + 18X^{28}Y^{38} - X^{27}Y^{39} + 18X^{29}Y^{36} + 42X^{28}Y^{37} + X^{26}Y^{39} \\
& + X^{30}Y^{34} - 3X^{29}Y^{35} - 31X^{28}Y^{36} - 15X^{27}Y^{37} - 5X^{26}Y^{38} - 23X^{28}Y^{35} + 29X^{27}Y^{36} + 14X^{26}Y^{37} \\
& + X^{29}Y^{33} + 2X^{28}Y^{34} + 17X^{27}Y^{35} - 5X^{25}Y^{37} + 3X^{28}Y^{33} - 30X^{27}Y^{34} - 20X^{26}Y^{35} + 5X^{25}Y^{36} \\
& + 3X^{28}Y^{32} - X^{27}Y^{33} + 20X^{26}Y^{34} + 33X^{25}Y^{35} - 2X^{24}Y^{36} - 12X^{26}Y^{33} - 20X^{25}Y^{34} - 9X^{24}Y^{35} \\
& - X^{27}Y^{31} - 4X^{26}Y^{32} - 34X^{25}Y^{33} + 15X^{24}Y^{34} + 2X^{23}Y^{35} + 3X^{27}Y^{30} + 12X^{26}Y^{31} + 9X^{25}Y^{32} \\
& + 34X^{24}Y^{33} + X^{23}Y^{34} - 7X^{26}Y^{30} + X^{25}Y^{31} - 48X^{24}Y^{32} - 17X^{23}Y^{33} + X^{22}Y^{34} + X^{26}Y^{29} \\
& + 13X^{25}Y^{30} - 14X^{24}Y^{31} + 18X^{23}Y^{32} + 10X^{22}Y^{33} + 4X^{25}Y^{29} + 13X^{24}Y^{30} - 4X^{23}Y^{31} \\
& + 5X^{22}Y^{32} - 5X^{25}Y^{28} - 4X^{24}Y^{29} - 20X^{23}Y^{30} - 36X^{22}Y^{31} - 3X^{21}Y^{32} + 15X^{24}Y^{28} \\
& + 20X^{23}Y^{29} + 3X^{22}Y^{30} + 26X^{21}Y^{31} + X^{20}Y^{32} - 4X^{24}Y^{27} - 8X^{23}Y^{28} - 2X^{22}Y^{29} \\
& - 21X^{21}Y^{30} - 3X^{20}Y^{31} + 22X^{22}Y^{28} - 22X^{21}Y^{29} + 3X^{23}Y^{26} + 21X^{22}Y^{27} + 2X^{21}Y^{28} \\
& + 8X^{20}Y^{29} + 4X^{19}Y^{30} - X^{23}Y^{25} - 26X^{22}Y^{26} - 3X^{21}Y^{27} - 20X^{20}Y^{28} - 15X^{19}Y^{29} \\
& + 3X^{22}Y^{25} + 36X^{21}Y^{26} + 20X^{20}Y^{27} + 4X^{19}Y^{28} + 5X^{18}Y^{29} - 5X^{21}Y^{25} + 4X^{20}Y^{26} \\
& - 13X^{19}Y^{27} - 4X^{18}Y^{28} - 10X^{21}Y^{24} - 18X^{20}Y^{25} + 14X^{19}Y^{26} - 13X^{18}Y^{27} - X^{17}Y^{28} \\
& - X^{21}Y^{23} + 17X^{20}Y^{24} + 48X^{19}Y^{25} - X^{18}Y^{26} + 7X^{17}Y^{27} - X^{20}Y^{23} - 34X^{19}Y^{24} \\
& - 9X^{18}Y^{25} - 12X^{17}Y^{26} - 3X^{16}Y^{27} - 2X^{20}Y^{22} - 15X^{19}Y^{23} + 34X^{18}Y^{24} + 4X^{17}Y^{25} \\
& + X^{16}Y^{26} + 9X^{19}Y^{22} + 20X^{18}Y^{23} + 12X^{17}Y^{24} + 2X^{19}Y^{21} - 33X^{18}Y^{22} - 20X^{17}Y^{23} \\
& + X^{16}Y^{24} - 3X^{15}Y^{25} - 5X^{18}Y^{21} + 20X^{17}Y^{22} + 30X^{16}Y^{23} - 3X^{15}Y^{24} + 5X^{18}Y^{20} \\
& - 17X^{16}Y^{22} - 2X^{15}Y^{23} - X^{14}Y^{24} - 14X^{17}Y^{20} - 29X^{16}Y^{21} + 23X^{15}Y^{22} + 5X^{17}Y^{19} \\
& + 15X^{16}Y^{20} + 31X^{15}Y^{21} + 3X^{14}Y^{22} - X^{13}Y^{23} - X^{17}Y^{18} - 42X^{15}Y^{20} - 18X^{14}Y^{21} \\
& + X^{16}Y^{18} - 18X^{15}Y^{19} + 26X^{14}Y^{20} + 13X^{13}Y^{21} + 19X^{15}Y^{18} + 10X^{14}Y^{19} - 2X^{13}Y^{20} \\
& - 2X^{15}Y^{17} - 32X^{14}Y^{18} - 32X^{13}Y^{19} - 2X^{12}Y^{20} + 2X^{15}Y^{16} + 9X^{14}Y^{17} + 9X^{13}Y^{18} \\
& + 30X^{12}Y^{19} + 3X^{11}Y^{20} + 14X^{13}Y^{17} - 19X^{12}Y^{18} - 8X^{11}Y^{19} + X^{14}Y^{15} - 6X^{13}Y^{16} \\
& - 40X^{12}Y^{17} + 7X^{11}Y^{18} + 2X^{13}Y^{15} + 21X^{12}Y^{16} + 16X^{11}Y^{17} + 2X^{10}Y^{18} - X^{13}Y^{14} + 8X^{12}Y^{15} \\
& - 19X^{11}Y^{16} - 10X^{10}Y^{17} + 4X^{12}Y^{14} - 10X^{11}Y^{15} + 2X^{10}Y^{16} + 4X^9Y^{17} - 3X^{12}Y^{13} \\
& + 6X^{11}Y^{14} + 11X^{10}Y^{15} - 7X^9Y^{16} + 2X^{11}Y^{13} - 4X^{10}Y^{14} - 13X^9Y^{15} + 2X^8Y^{16} + X^{11}Y^{12} \\
& + 6X^{10}Y^{13} + 4X^9Y^{14} + X^8Y^{15} - 4X^{10}Y^{12} + 9X^9Y^{13} - 6X^8Y^{14} - X^{10}Y^{11} + 6X^9Y^{12} - 8X^8Y^{13} \\
& + X^7Y^{14} - X^9Y^{11} + 6X^8Y^{12} + 3X^7Y^{13} - 3X^9Y^{10} + 4X^8Y^{11} - 4X^7Y^{12} - 2X^6Y^{13} + 2X^8Y^{10} \\
& + X^7Y^{11} + X^6Y^{12} - X^8Y^9 + 2X^7Y^{10} + 6X^6Y^{11} - 4X^7Y^9 - 3X^5Y^{11} + X^7Y^8 + 3X^6Y^9 + 3X^5Y^{10}
\end{aligned}$$

$$\begin{aligned}
& -5X^6Y^8 + X^5Y^9 - X^4Y^{10} - 3X^6Y^7 + 3X^5Y^8 + 2X^4Y^9 + X^6Y^6 + 4X^5Y^7 - 4X^5Y^6 - 5X^4Y^7 \\
& + 3X^4Y^6 + 3X^3Y^7 + X^4Y^5 - 4X^3Y^6 - X^2Y^7 - 4X^3Y^5 + 2X^2Y^6 + 3X^3Y^4 + X^2Y^5 - 4X^2Y^4 + 1
\end{aligned}$$

B. Formulae for local graded subalgebra zeta functions

$$\begin{aligned}
W_{531} = & (-X^5Y^{18} - X^5Y^{16} - X^5Y^{15} - X^4Y^{16} - X^5Y^{14} - X^4Y^{15} + 2X^4Y^{13} + X^3Y^{14} \\
& + X^4Y^{12} + 2X^3Y^{13} + X^4Y^{11} + X^3Y^{12} + X^2Y^{13} + X^4Y^{10} + X^3Y^{11} + X^4Y^9 \\
& + 3X^3Y^{10} + 2X^3Y^9 + X^2Y^{10} - X^3Y^8 - 2X^2Y^9 - 3X^2Y^8 - XY^9 - X^2Y^7 \\
& - XY^8 - X^3Y^5 - X^2Y^6 - XY^7 - 2X^2Y^5 - XY^6 - X^2Y^4 - 2XY^5 + XY^3 \\
& + Y^4 + XY^2 + Y^3 + Y^2 + 1) \\
& /((1 - XY^5)(1 - X^2Y^3)(1 - X^2Y^4)(1 - XY^2)(1 - XY)(1 - Y^5)(1 - Y^2)(1 - Y))
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
W_{541} = & (-X^3Y^{21} - X^3Y^{20} - 3X^3Y^{19} - 5X^3Y^{18} - 7X^3Y^{17} + X^2Y^{18} - 8X^3Y^{16} + 4X^2Y^{17} \\
& - 7X^3Y^{15} + 9X^2Y^{16} - 6X^3Y^{14} + 16X^2Y^{15} - 6X^3Y^{13} + 19X^2Y^{14} - XY^{15} - 4X^3Y^{12} \\
& + 21X^2Y^{13} - 4XY^{14} - 3X^3Y^{11} + 21X^2Y^{12} - 8XY^{13} - X^3Y^{10} + 20X^2Y^{11} - 14XY^{12} \\
& + 18X^2Y^{10} - 18XY^{11} + 14X^2Y^9 - 20XY^{10} + Y^{11} + 8X^2Y^8 - 21XY^9 + 3Y^{10} \\
& + 4X^2Y^7 - 21XY^8 + 4Y^9 + X^2Y^6 - 19XY^7 + 6Y^8 - 16XY^6 + 6Y^7 - 9XY^5 \\
& + 7Y^6 - 4XY^4 + 8Y^5 - XY^3 + 7Y^4 + 5Y^3 + 3Y^2 + Y + 1) \\
& /((1 - XY^4)(1 - XY^3)(1 - XY^2)(1 - XY)(1 - Y^7)(1 - Y^4)(1 - Y^3)(1 - Y^2))
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
W_{621} = & (X^4Y^8 + X^4Y^6 + X^3Y^6 - X^3Y^5 + X^2Y^6 - X^3Y^4 - X^2Y^5 - X^3Y^3 - XY^5 \\
& - X^2Y^3 - XY^4 + X^2Y^2 - XY^3 + XY^2 + Y^2 + 1)(XY^2 + 1) \\
& /((1 - X^2Y^3)(1 - XY^3)(1 - X^3Y^2)(1 - X^2Y^2)(1 - X^2Y)(1 - XY)(1 - Y^3)(1 - Y))
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
W_{623} = & (X^9Y^{14} + X^9Y^{13} + X^9Y^{12} - 3X^8Y^{11} - 2X^8Y^{10} - X^6Y^{11} - X^6Y^{10} + X^7Y^8 - 3X^6Y^9 \\
& - X^5Y^9 - X^4Y^{10} + X^6Y^7 + 2X^5Y^8 + X^6Y^6 + 2X^5Y^7 + 2X^4Y^8 + 2X^5Y^6 + 2X^4Y^7 \\
& + X^3Y^8 + 2X^4Y^6 + X^3Y^7 - X^5Y^4 - X^4Y^5 - 3X^3Y^5 + X^2Y^6 - X^3Y^4 - X^3Y^3 \\
& - 2XY^4 - 3XY^3 + Y^2 + Y + 1) / ((1 - X^4Y^3)(1 - X^3Y^3)(1 - XY^3)(1 - XY^2)^2 \\
& \quad \times (1 - X^3Y)(1 - X^2Y)(1 - XY)(1 - Y^2)^2)
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
W_{631} = & (-X^5Y^{18} - X^5Y^{16} - X^5Y^{15} - X^4Y^{16} - X^5Y^{14} - X^4Y^{15} + 2X^4Y^{13} + X^3Y^{14} \\
& + X^4Y^{12} + 2X^3Y^{13} + X^4Y^{11} + X^3Y^{12} + X^2Y^{13} + X^4Y^{10} + X^3Y^{11} + X^4Y^9 \\
& + 3X^3Y^{10} + 2X^3Y^9 + X^2Y^{10} - X^3Y^8 - 2X^2Y^9 - 3X^2Y^8 - XY^9 - X^2Y^7 - XY^8 \\
& - X^3Y^5 - X^2Y^6 - XY^7 - 2X^2Y^5 - XY^6 - X^2Y^4 - 2XY^5 + XY^3 + Y^4 + XY^2 \\
& + Y^3 + Y^2 + 1) / ((1 - XY^5)(1 - X^2Y^4)(1 - X^2Y^3)(1 - XY^2)(1 - X^2Y) \\
& \quad \times (1 - XY)(1 - Y^5)(1 - Y^2)(1 - Y))
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
W_{632} = & (X^3Y^{16} + 2X^3Y^{15} + 4X^3Y^{14} + 7X^3Y^{13} + X^2Y^{14} + 10X^3Y^{12} + 2X^2Y^{13} + 11X^3Y^{11} \\
& - XY^{13} + 10X^3Y^{10} - 3X^2Y^{11} - 3XY^{12} + 7X^3Y^9 - 8X^2Y^{10} - 6XY^{11} + 4X^3Y^8 \\
& - 11X^2Y^9 - 9XY^{10} + 2X^3Y^7 - 11X^2Y^8 - 10XY^9 + Y^{10} + X^3Y^6 - 10X^2Y^7 \\
& - 11XY^8 + 2Y^9 - 9X^2Y^6 - 11XY^7 + 4Y^8 - 6X^2Y^5 - 8XY^6 + 7Y^7 - 3X^2Y^4 \\
& - 3XY^5 + 10Y^6 - X^2Y^3 + 11Y^5 + 2XY^3 + 10Y^4 + XY^2 + 7Y^3 + 4Y^2 + 2Y + 1) \\
& \times (1 - Y) / ((1 - XY^3)(1 - X^2Y^2)(1 - XY^2)(1 - X^2Y)(1 - XY) \\
& \quad \times (1 - Y^5)(1 - Y^4)(1 - Y^3)(1 - Y^2))
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
W_{633} = & (X^3Y^{14} + X^3Y^{13} + 3X^3Y^{12} + 4X^3Y^{11} + X^2Y^{12} + 5X^3Y^{10} + 4X^3Y^9 - X^2Y^{10} \\
& - XY^{11} + 3X^3Y^8 - 3X^2Y^9 - XY^{10} + X^3Y^7 - 5X^2Y^8 - 4XY^9 + X^3Y^6 - 5X^2Y^7 \\
& - 4XY^8 - 4X^2Y^6 - 5XY^7 + Y^8 - 4X^2Y^5 - 5XY^6 + Y^7 - X^2Y^4 - 3XY^5 + 3Y^6 \\
& - X^2Y^3 - XY^4 + 4Y^5 + 5Y^4 + XY^2 + 4Y^3 + 3Y^2 + Y + 1) \\
& / ((1 - XY^3)(1 - X^2Y^2)(1 - XY^2)(1 - X^2Y)(1 - XY)(1 - Y^3)(1 - Y^4)^2)
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
W_{643} = & (-X^3Y^{21} - X^3Y^{20} - 3X^3Y^{19} - 5X^3Y^{18} - 7X^3Y^{17} + X^2Y^{18} - 8X^3Y^{16} + 4X^2Y^{17} \\
& - 7X^3Y^{15} + 9X^2Y^{16} - 6X^3Y^{14} + 16X^2Y^{15} - 6X^3Y^{13} + 19X^2Y^{14} - XY^{15} - 4X^3Y^{12} \\
& + 21X^2Y^{13} - 4XY^{14} - 3X^3Y^{11} + 21X^2Y^{12} - 8XY^{13} - X^3Y^{10} + 20X^2Y^{11} - 14XY^{12} \\
& + 18X^2Y^{10} - 18XY^{11} + 14X^2Y^9 - 20XY^{10} + Y^{11} + 8X^2Y^8 - 21XY^9 + 3Y^{10} \\
& + 4X^2Y^7 - 21XY^8 + 4Y^9 + X^2Y^6 - 19XY^7 + 6Y^8 - 16XY^6 + 6Y^7 - 9XY^5 + 7Y^6 \\
& - 4XY^4 + 8Y^5 - XY^3 + 7Y^4 + 5Y^3 + 3Y^2 + Y + 1) \\
& / ((1 - XY^4)(1 - XY^3)(1 - XY^2)(1 - X^2Y)(1 - XY) \\
& \quad \times (1 - Y^7)(1 - Y^4)(1 - Y^3)(1 - Y^2))
\end{aligned} \tag{B.8}$$

References

- [1] W. W. Adams and P. Lounstaunau, *An introduction to Gröbner bases*, Graduate Studies in Mathematics, vol. 3, American Mathematical Society, Providence, RI, 1994.
- [2] N. Avni, B. Klopsch, U. Onn, and C. Voll, *Representation zeta functions of compact p -adic analytic groups and arithmetic groups*, *Duke Math. J.* **162** (2013), no. 1, 111–197.
- [3] V. Baldoni, N. Berline, J. A. De Loera, B. Dutra, M. Köppe, S. Moreinis, G. Pinto, M. Vergne, and J. Wu, *A user's guide for LattE integrale v1.7.3*, 2015. Software package. LattE is available at <http://www.math.ucdavis.edu/~latte/>.
- [4] A. Barvinok, *Integer points in polyhedra*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.
- [5] A. Barvinok and J. E. Pommersheim, *An algorithmic theory of lattice points in polyhedra*, *New perspectives in algebraic combinatorics* (Berkeley, CA, 1996–97), 1999, pp. 91–147.
- [6] A. Barvinok and K. Woods, *Short rational generating functions for lattice point problems*, *J. Amer. Math. Soc.* **16** (2003), no. 4, 957–979 (electronic).
- [7] A. I. Barvinok, *A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed*, *Math. Oper. Res.* **19** (1994), no. 4, 769–779.
- [8] W. A. de Graaf, *Classification of solvable Lie algebras*, *Experiment. Math.* **14** (2005), no. 1, 15–25.
- [9] W. A. de Graaf, *Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2*, *J. Algebra* **309** (2007), no. 2, 640–653.
- [10] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice it Corps de classes local par Michiel Hazewinkel.
- [11] J. Denef, *Report on Igusa's local zeta function*, *Astérisque* **201-203** (1991), Exp. No. 741, 359–386 (1992). Séminaire Bourbaki, Vol. 1990/91.
- [12] J. Denef and F. Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, *J. Amer. Math. Soc.* **5** (1992), no. 4, 705–720.
- [13] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 7.3)*, 2016. Available from <http://www.sagemath.org/>.
- [14] M. P. F. du Sautoy, *The zeta function of $\mathfrak{sl}_2(\mathbb{Z})$* , *Forum Math.* **12** (2000), no. 2, 197–221.
- [15] M. P. F. du Sautoy and F. J. Grunewald, *Analytic properties of zeta functions and subgroup growth*, *Ann. of Math. (2)* **152** (2000), no. 3, 793–833.
- [16] M. P. F. du Sautoy and F. Loeser, *Motivic zeta functions of infinite-dimensional Lie algebras*, *Selecta Math. (N.S.)* **10** (2004), no. 2, 253–303.
- [17] M. P. F. du Sautoy and G. Taylor, *The zeta function of \mathfrak{sl}_2 and resolution of singularities*, *Math. Proc. Cambridge Philos. Soc.* **132** (2002), no. 1, 57–73.
- [18] M. P. F. du Sautoy and L. Woodward, *Zeta functions of groups and rings*, *Lecture Notes in Mathematics*, vol. 1925, Springer-Verlag, Berlin, 2008.
- [19] D. H. Dung and C. Voll, *Uniform analytic properties of representation zeta functions of finitely generated nilpotent groups*, *Trans. Amer. Math. Soc.* **369** (2017), no. 9, 6327–6349.
- [20] A. Evseev, *Reduced zeta functions of Lie algebras*, *J. Reine Angew. Math.* **633** (2009), 197–211.
- [21] S. Ezzat, *Representation growth of finitely generated torsion-free nilpotent groups: Methods and examples*. PhD Thesis, University of Canterbury, 2012. See <http://hdl.handle.net/10092/7235>.
- [22] ———, *Counting irreducible representations of the Heisenberg group over the integers of a quadratic number field*, *J. Algebra* **397** (2014), 609–624.
- [23] E. S. Golod and I. R. Šafarevič, *On the class field tower (Russian)*, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 261–272.

- [24] J. González-Sánchez, A. Jaikin-Zapirain, and B. Klopsch, *The representation zeta function of a FAB compact p -adic Lie group vanishes at -2* , Bull. Lond. Math. Soc. **46** (2014), no. 2, 239–244.
- [25] G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR 3.0, Centre for Computer Algebra, University of Kaiserslautern, 2005. Available from <http://www.singular.uni-kl.de/>.
- [26] F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), no. 1, 185–223.
- [27] E. Hrushovski, B. Martin, S. Rideau, and R. Cluckers, *Definable equivalence relations and zeta functions of groups* (2017). To appear in J. Eur. Math. Soc. (JEMS).
- [28] I. Ilani, *Zeta functions related to the group $SL_2(\mathbb{Z}_p)$* , Israel J. Math. **109** (1999), 157–172.
- [29] N. M. Katz, *Review of ℓ -adic cohomology*, Motives (Seattle, WA, 1991), 1994, pp. 21–30.
- [30] B. Klopsch and C. Voll, *Zeta functions of three-dimensional p -adic Lie algebras*, Math. Z. **263** (2009), no. 1, 195–210.
- [31] O. Kuzmich, *Graded nilpotent Lie algebras in low dimensions*, Lobachevskii J. Math. **3** (1999), 147–184 (electronic). Towards 100 years after Sophus Lie (Kazan, 1998).
- [32] V. V. Morozov, *Classification of nilpotent Lie algebras of sixth order*, Izv. Vysš. Učebn. Zaved. Matematika **1958** (1958), no. 4 (5), 161–171.
- [33] J. Nakagawa, *Orders of a quartic field*, Mem. Amer. Math. Soc. **122** (1996), no. 583, viii+75.
- [34] C. Nunley and A. Magid, *Simple representations of the integral Heisenberg group*, Classical groups and related topics (Beijing, 1987), 1989, pp. 89–96.
- [35] T. Rossmann, *Computing topological zeta functions of groups, algebras, and modules, I*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 5, 1099–1134.
- [36] ———, *Computing topological zeta functions of groups, algebras, and modules, II*, J. Algebra **444** (2015), 567–605.
- [37] ———, *Topological representation zeta functions of unipotent groups*, J. Algebra **448** (2016), 210–237.
- [38] ———, *Enumerating submodules invariant under an endomorphism*, Math. Ann. **368** (2017), no. 1-2, 391–417.
- [39] ———, *Stability results for local zeta functions of groups algebras, and modules*, Math. Proc. Cambridge Philos. Soc. (2017), 1–10. DOI:10.1017/S0305004117000585.
- [40] ———, *Zeta, version 0.3.2*, 2017. See <http://www.math.uni-bielefeld.de/~rossmann/Zeta/>.
- [41] J.-P. Serre, *Lectures on $N_X(p)$* , Chapman & Hall/CRC Research Notes in Mathematics, vol. 11, CRC Press, Boca Raton, FL, 2012.
- [42] R. Snocken, *Zeta functions of groups and rings*. PhD Thesis, University of Southampton, 2012. See <http://eprints.soton.ac.uk/id/eprint/372833>.
- [43] L. Solomon, *Zeta functions and integral representation theory*, Advances in Math. **26** (1977), no. 3, 306–326.
- [44] A. Stasinski and C. Voll, *Representation zeta functions of nilpotent groups and generating functions for Weyl groups of type B* , Amer. J. Math. **136** (2014), no. 2, 501–550.
- [45] G. Taylor, *Zeta functions of algebras and resolution of singularities*, Ph.D. Thesis, 2001.
- [46] C. Voll, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), no. 2, 1181–1218.
- [47] ———, *Local functional equations for submodule zeta functions associated to nilpotent algebras of endomorphisms* (2016). To appear in Int. Math. Res. Not.
- [48] J. White, *Zeta functions of groups*. DPhil Thesis, University of Oxford, 2000.
- [49] K. Woods and R. Yoshida, *Short rational generating functions and their applications to integer programming*, SIAG/OPT Views-and-News **16** (2005), no. 1–2, 15–19.

- [50] L. Woodward, *Zeta functions of groups: computer calculations and functional equations*. DPhil Thesis, University of Oxford, 2005.
- [51] ———, *Zeta functions of Lie rings of upper-triangular matrices*, J. Lond. Math. Soc. (2) **77** (2008), no. 1, 69–82.