Reduced and topological zeta functions in enumerative algebra

TOBIAS ROSSMANN

Setup. Let $(a_n)_{n\geq 1}$ be a sequence of numbers attached to an instance of an algebraic counting problem. Classical examples are obtained by letting a_n denote the number of subgroups or subalgebras of index n in a given group G or algebra A, respectively, or by taking a_n to be the number of irreducible representations $G \to \mathrm{GL}_n(\mathbf{C})$, counted up to suitable equivalence. A common theme in enumerative algebra is to study such sequences by means of the associated *global zeta function* $\mathsf{Z}(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Under suitable assumptions, this series will admit an Euler product decomposition $\mathsf{Z}(s) = \prod_p \mathsf{Z}_p(s)$, where p ranges over primes and the *local zeta function* at p is $\mathsf{Z}_p(s) = \sum_{k=0}^{\infty} a_{p^k} p^{-ks}$. Subject to further assumptions on the shapes of the $\mathsf{Z}_p(s)$, the reduced and topological zeta functions $\mathsf{Z}_{\mathrm{red}}(T)$ and $\mathsf{Z}_{\mathrm{top}}(s)$ are two related but subtly different rational functions obtained by taking limits of $\mathsf{Z}_p(s)$ as "$p \to 1$".

Topological zeta functions. Informally, the topological zeta function $\mathsf{Z}_{\mathrm{top}}(s) \in \mathbf{Q}(s)$ is the constant term of $(1-p^{-1})^e \mathsf{Z}_p(s)$ as a power series in $p-1$. Here, the exponent $e \in \mathbf{Z}$ depends both on the counting problem and on the particular instance. In a surprising number of cases of interest, the local zeta functions $\mathsf{Z}_p(s)$ are *(almost) uniform* in the sense that there exists a single rational function $W(X,T) \in \mathbf{Q}(X,T)$ such that $\mathsf{Z}_p(s) = W(p, p^{-s})$ for (almost) all primes p. In such cases, we formally expand $W(p,p^{-s})$ using the binomial series $p^{-s} = (1 + (p-1))^{-s} = \sum_{m=0}^{\infty} \binom{-s}{m}(p-1)^m$ and we obtain $\mathsf{Z}_{\mathrm{top}}(s)$ as indicated above.

The key difficulty in rigorously defining topological zeta functions is to overcome the restriction to (almost) uniform cases. The dependence of $\mathsf{Z}_p(s)$ on p is often governed by a formula of a type that first appeared in work of Denef [1, §3]. In these cases, there are schemes $V_1, \dots, V_r$ (over $\mathbf{Z}$) and rational functions $W_1(X,T), \dots, W_r(X,T)$ such that for almost all primes p, we have $\mathsf{Z}_p(s) = \sum_{i=1}^{r} \#V_i(\mathbf{F}_p) W_i(p,p^{-s})$. Using such a formula, Denef and Loeser [2] gave a rigorous definition of the topological zeta functions associated with a polynomial. The functions $W_i(p,p^{-s})$ are again expanded using the binomial series. Using ℓ-adic interpolation arguments based on Grothendieck's trace formula, the limit of $\#V_i(\mathbf{F}_p)$ as "$p \to 1$" is $\chi(V_i(\mathbf{C}))$, the topological Euler characteristic of the complex analytic space attached to V_i. By [10, §3], in almost uniform cases, our informal approach agrees with the rigorous one.

In [3], Denef and Loeser gave another, independent description of the topological zeta function associated with a polynomial by means of a suitable specialisation of the associated *motivic zeta function*. In [5, §8], du Sautoy and Loeser defined topological subalgebra zeta functions by specialising motivic ones, having introduced the latter in the same paper. An ℓ-adic approach to topological subobject zeta functions based on [2] was developed in [8, §5].

Global and local subalgebra zeta functions. Subalgebra zeta functions were introduced by Grunewald, Segal, and Smith [7, §3]. For the remainder of this abstract, let A be a (not necessarily associative) $\mathbf{Z}$-algebra whose underlying $\mathbf{Z}$-module is free of finite rank d. Let $a_n(A)$ denote the number of subalgebras B of A of additive index $|A : B| = n$. Let $\mathsf{Z}^A(s) = \sum_{n=1}^{\infty} a_n(A) n^{-s}$ be the associated *(global) subalgebra zeta function*. By the Chinese remainder theorem, we obtain an Euler product $\mathsf{Z}^A(s) = \prod_p \mathsf{Z}_p^A(s)$ as above. The *local subalgebra zeta function* $\mathsf{Z}_p^A(s)$ enumerates subalgebras of the $\mathbf{Z}_p$-algebra $A \otimes \mathbf{Z}_p$. In [4], du Sautoy and Grunewald established Denef-style formulae for local subalgebra zeta functions associated with a fixed algebra A. Voll [12, Thm A] established a local functional equation of subalgebra zeta functions under "inversion of p" of the form

$$(\star) \qquad \mathsf{Z}_p^A(s)\Big|_{p \leftarrow p^{-1}} = (-1)^d p^{\binom{d}{2} - ds} \cdot \mathsf{Z}_p^A(s).$$

Voll's proof relied on a delicate interplay of (a) the functional equations satisfied by Hasse-Weil zeta functions of smooth projective varieties and (b) a self-reciprocity property of generating functions enumerating lattice points within certain cones. Part (a) is explained by Poincaré duality in ℓ-adic cohomology while Stanley [11, Ch. I] elegantly explained (b) in terms of local cohomology.

Reduced and topological subalgebra zeta functions. Using our informal approach, the *topological subalgebra zeta function* $\mathsf{Z}_{\mathrm{top}}^A(s) \in \mathbf{Q}(s)$ is the constant term of $(1-p^{-1})^d \mathsf{Z}_p^A(s)$ as a series in $p-1$. Introduced by Evseev [6], the *reduced subalgebra zeta function* $\mathsf{Z}_{\mathrm{red}}^A(T) \in \mathbf{Q}[\![T]\!] \cap \mathbf{Q}(T)$ is obtained by viewing $\mathsf{Z}_p^A(s)$ as a series in $T = p^{-s}$ and applying a limit "$p \to 1$" to its coefficients. In (almost) uniform cases in which $\mathsf{Z}_p^A(s) = W(p, p^{-s})$ for (almost) all p, the reduced subalgebra zeta function is given by $\mathsf{Z}_{\mathrm{red}}^A(T) = W(1, T)$.

Upon taking the limit "$p \to 1$", Voll's local functional equation $(\star)$ implies that $\mathsf{Z}_{\mathrm{red}}^A(T)$ satisfies the self-reciprocity identity $\mathsf{Z}_{\mathrm{red}}^A(T^{-1}) = (-1)^d T^d \cdot \mathsf{Z}_{\mathrm{red}}^A(T)$, a property reminiscent of Hilbert series of graded Gorenstein algebras.

Conjectures. Reduced and topological zeta functions are seemingly quite different invariants. Both constructions are, however, conjecturally related by the "coincidence conjecture" below. We first recall the following conjecture which predicts the vanishing order of topological subalgebra zeta functions at infinity.

Degree conjecture ([8, Conj. I])**.** $\deg(\mathsf{Z}_{\mathrm{top}}^A(s)) = -d$.

Evseev [6, Prop. 4.1] showed that if A admits a particular type of basis, then there exists a d-dimensional cone $\mathcal{C} \subset \mathbf{R}_{\geq 0}^d$ such that the reduced zeta function $\mathsf{Z}_{\mathrm{red}}^A(T)$ is the (coarse) Hilbert series of the affine monoid algebra $\mathbf{Q}[\mathcal{C} \cap \mathbf{Z}^d]$. Using the explicit description of $\mathcal{C}$ by Evseev, results from combinatorial commutative algebra (see [11, Ch. 1]) show that $\mathbf{Q}[\mathcal{C} \cap \mathbf{Z}^d]$ is Gorenstein.

Hilbert series conjecture (Voll)**.** *There exists a (natural, meaningful) $\mathbf{N}_0$-graded Gorenstein algebra of dimension d whose Hilbert series is $\mathsf{Z}_{\mathrm{red}}^A(T)$.*

While Voll's Hilbert series conjecture has been informally shared with researchers for quite some time, it took more cautious forms in the published literature. Inspired by the preceding two conjectures, define

$$\mathrm{m}_{\mathrm{top}}(A) := s^{-d}\mathsf{Z}^{A}_{\mathrm{top}}(s^{-1})\Big|_{s=0}, \quad \mathrm{m}_{\mathrm{red}}(A) := (1-T)^{d}\mathsf{Z}^{A}_{\mathrm{red}}(T)\Big|_{T=1}.$$

The topological degree conjecture is equivalent to $\mathrm{m}_{\mathrm{top}}(A)$ being nonzero and finite. Similarly, if the Hilbert series conjecture is true, then $\mathsf{Z}^{A}_{\mathrm{red}}(T)$ has a pole of order d at $T = 1$ whence $\mathrm{m}_{\mathrm{red}}(A)$ is nonzero and finite.

Coincidence conjecture. $\mathrm{m}_{\mathrm{red}}(A) = \mathrm{m}_{\mathrm{top}}(A)$ *and the common value is a positive rational number.*

While the preceding conjecture has been informally communicated for at least a decade, to the best of the author's knowledge, it too has not, so far, been formally stated as such in a published document. This notwithstanding, apart from the numerical evidence provided by computer calculations [8, 9], the coincidence conjecture has been verified for some families of algebras; see e.g. [13, §3.4].

References

[1] J. Denef, *On the degree of Igusa's local zeta function*, Amer. J. Math. **109** (1987), no. 6, 991–1008.

[2] J. Denef and F. Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc. **5** (1992), no. 4, 705–720.

[3] J. Denef and F. Loeser, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), no. 3, 505–537.

[4] M. P. F. du Sautoy and F. Grunewald, *Analytic properties of zeta functions and subgroup growth*, Ann. Math. (2) **152** (2000), no. 3, 793–833.

[5] M. P. F. du Sautoy and F. Loeser, *Motivic zeta functions of infinite-dimensional Lie algebras*, Selecta Math. (N.S.) **10** (2004), no. 2, 253–303.

[6] A. Evseev, *Reduced zeta functions of Lie algebras*, J. Reine Angew. Math. 633 (2009), 197–211.

[7] F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. 93 (1988), no. 1, 185–223.

[8] T. Rossmann, *Computing topological zeta functions of groups, algebras, and modules I*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 5, 1099–1134.

[9] T. Rossmann, *Computing topological zeta functions of groups, algebras, and modules, II*, J. Algebra **444** (2015), 567–605.

[10] T. Rossmann, *Stability results for local zeta functions of groups, algebras, and modules*, Math. Proc. Camb. Phil. Soc. **165** (2018), no. 3, 435–444.

[11] R. P. Stanley, *Combinatorics and commutative algebra.* Progr. Math., 41. Birkhäuser Boston, Inc., Boston, MA, 1983. viii+88 pp

[12] C. Voll, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), no. 2, 1181–1218.

[13] C. Voll, *Ideal zeta functions associated to a family of class-2-nilpotent Lie rings*, Q. J. Math. **71** (2020), no. 3, 959–980.