Orbits: tame and wild

Tobias Rossmann

This talk was devoted to recent developments in the symbolic enumeration of orbits. Let $\mathbf{G}$ be a group scheme acting on a scheme $\mathbf{X}$. (Schemes are assumed to be separated and of finite type in the following.) For example, $\mathbf{G} \leqslant \mathrm{GL}_n$ could be a linear group scheme acting naturally on affine n-space or on itself by conjugation. Our goal is to determine the number of orbits $h(\mathbf{G} \curvearrowright \mathbf{X}; q) := |\mathbf{X}(\mathbf{F}_q)/\mathbf{G}(\mathbf{F}_q)|$ as a "symbolic function" of the prime power q. Let $\mathbf{Y}$ be the scheme representing the functor $R \rightsquigarrow \{(x, g) \in \mathbf{X}(R) \times \mathbf{G}(R) : xg = x\}$. Burnside's lemma shows that $|\mathbf{X}(\mathbf{F}_q)/\mathbf{G}(\mathbf{F}_q)| = |\mathbf{Y}(\mathbf{F}_q)|/|\mathbf{G}(\mathbf{F}_q)|$. Hence, $h(\mathbf{G} \curvearrowright \mathbf{X}; q)$ is expressible in terms of the numbers of $\mathbf{F}_q$-rational points of schemes—this is an archetypal example of a function which depends geometrically on q. We regard such functions as *tame* when they are (close to being) polynomial in q; otherwise, they are *wild*.

It is natural to ask just how wild the functions $q \mapsto h(\mathbf{G} \curvearrowright \mathbf{X}; q)$ can be if $\mathbf{G}$ and $\mathbf{X}$ are themselves restricted to be geometrically tame. By [8, Thm A], we can approximate (in a suitable sense) the number of $\mathbf{F}_q$-rational points on an arbitrary scheme by means of (a) numbers of linear orbits of commutative unipotent groups or (b) by means of class numbers of Baer group schemes, all uniformly in q. This combines a deep result due to Belkale and Brosnan [1] and recent techniques surrounding so-called *ask zeta functions* (introduced in [5]). We conclude that symbolically enumerating linear orbits and conjugacy classes of unipotent groups is a fundamentally hopeless task in the sense that it is as hard as enumerating solutions to arbitrary $\mathbf{Z}$-defined systems of polynomial equations over $\mathbf{F}_q$.

On the other hand, for many specific families of groups of interest, we can of course do much better. Let $\mathrm{k}(H)$ denote the number of conjugacy classes ("class number") of a group H. Let $\mathfrak{O}$ be a compact discrete valuation ring with maximal ideal $\mathfrak{P}$, e.g. the p-adic integers $\mathbf{Z}_p$ or a power series ring $\mathbf{F}_q[\![z]\!]$. The *class-counting zeta function* of a group scheme $\mathbf{G}$ over $\mathfrak{O}$ is the generating function

$$\mathsf{Z}^{\mathrm{cc}}_{\mathbf{G},\mathfrak{O}}(T) := \sum_{k=0}^{\infty} \mathrm{k}(\mathbf{G}(\mathfrak{O}/\mathfrak{P}^k))T^k.$$

The second half of the talk focused on recent developments surrounding class-counting zeta functions of *graphical groups*. Given a graph Γ and (commutative) ring R, the graphical group $\mathbf{G}_\Gamma(R)$ is a certain nilpotent group of class at most 2 whose commutator structure encodes adjacency in Γ. For precise definitions, see [7, §3.4] or [6, §1.1]. For example, the graphical group scheme associated with a complete graph K_n is a group scheme version of the free class-2-nilpotent group on n generators. By [7, Thm A], given any Γ, there exists an explicitly computable rational function $W_\Gamma(X, T) \in \mathbf{Q}(X, T)$ (denoted $W^-_\Gamma(X, T)$ in [7]) such that for each compact discrete valuation ring $\mathfrak{O}$ with residue field size q, we have $\mathsf{Z}^{\mathrm{cc}}_{\mathbf{G}_\Gamma,\mathfrak{O}}(T) = W_\Gamma(q, q^m T)$; here, m denotes the number of edges of Γ. We conclude that the class numbers $\mathrm{k}(\mathbf{G}_\Gamma(\mathfrak{O}/\mathfrak{P}^k))$ depend tamely on $\mathfrak{O}$ and also on the congruence level k.

In the talk, I reported on two further recent results that both establish forms of tameness with respect to natural graph-theoretic operations:

- The *join* $\Gamma_1 \vee \Gamma_2$ of graphs Γ_1 and Γ_2 is obtained from their disjoint union $\Gamma_1 \oplus \Gamma_2$ by adding an edge connecting each vertex of Γ_1 to each vertex of Γ_2. By [9, Thm A], the rational function $W_{\Gamma_1 \vee \Gamma_2}(X, T)$ is expressible as an explicit "distorted sum" of translates of $W_{\Gamma_1}(X, T)$ and $W_{\Gamma_2}(X, T)$. This result relies on a description of $W_\Gamma(X, T)$ in terms of p-adic integrals involving what we call animations. By an *animation* of a graph $\Gamma = (V, E)$, we mean a partial function from V to V which, whenever defined, sends a vertex to one of its neighbours.
- It is easy to see that $W_{\Gamma_1 \oplus \Gamma_2}(X, T)$ is the *Hadamard product* of $W_{\Gamma_1}(X, T)$ and $W_{\Gamma_2}(X, T)$. In general, predicting properties of Hadamard products of rational generating functions from properties of the factors seems to be very difficult. Building upon and extending work of Gessel and Zhuang [4], in [3] (see also [2]), we obtained explicit formulae for Hadamard products of certain rational generating functions. As an application, in [3, §5.3], we recorded several instances of explicit formulae for zeta functions enumerating linear orbits and conjugacy classes. In particular, our findings show that, given n, there exists an explicit rational function $W_n(X, Y_1, \ldots, Y_n, T)$ such that, up to an explicit translation, the rational function $W_{\mathrm{K}_{d_1} \oplus \cdots \oplus \mathrm{K}_{d_n}}(X, T)$ coincides with $W_n(X, X^{d_1}, \ldots, X^{d_n}, T)$.

Both results have direct applications to explicitly computing $W_\Gamma(X, T)$ and hence to symbolically enumerating conjugacy classes of graphical groups.

References

[1] P. Belkale and P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, Duke Math. J. 116 (2003), no. 1, 147–188.

[2] A. Carnevale, V. D. Moustakas, and T. Rossmann, *From coloured permutations to Hadamard products and zeta functions*. Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum). Sém. Lothar. Combin. 91B (2024), Article #56, 12 pp.

[3] A. Carnevale, V. D. Moustakas, and T. Rossmann, *Coloured shuffle compatibility, Hadamard products, and ask zeta functions*. Bull. Lond. Math. Soc. (2025), 23 pages. DOI:10.1112/blms.70081

[4] I. M. Gessel and Y. Zhuang, *Shuffle-compatible permutation statistics*. Adv. Math. 332 (2018), 85–141.

[5] T. Rossmann, *The average size of the kernel of a matrix and orbits of linear groups*. Proc. Lond. Math. Soc. (3) 117 (2018), no. 3, 574–616.

[6] T. Rossmann, *Enumerating conjugacy classes of graphical groups over finite fields*. Bull. Lond. Math. Soc. 54 (2022), no. 5, 1923–1943

[7] T. Rossmann and C. Voll, *Groups, graphs, and hypergraphs: average sizes of kernels of generic matrices with support constraints*. Mem. Amer. Math. Soc. 294 (2024), no. 1465, vi+120 pp.

[8] T. Rossmann, *On the enumeration of orbits of unipotent groups over finite fields*. Proc. Amer. Math. Soc. 153 (2025), no. 2, 479–495.

[9] T. Rossmann and C. Voll, *Ask zeta functions of joins of graphs* (preprint). arXiv:2505.10263