Towards a symbolic enumeration of orbits

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Let U_n be the group scheme of upper unitriangular $n \times n$ matrices. Let $\mathbf{G} \leqslant \mathrm{U}_n$ be a unipotent group scheme. Let $\mathfrak{O}$ be a compact discrete valuation ring with maximal ideal $\mathfrak{P}$. This talk was devoted to the enumeration of the orbits of $\mathbf{G}(\mathfrak{O}/\mathfrak{P}^m)$ on its natural module and to the enumeration of the conjugacy classes of $\mathbf{G}(\mathfrak{O}/\mathfrak{P}^m)$.

Restricting attention to unipotent groups provides us with a rich structure. Suppose that $\mathbf{G}$ is fixed and that the residue characteristic of $\mathfrak{O}$ is sufficiently large. For the enumeration of linear orbits, we can then assume that $\mathbf{G}$ is an abelian group scheme attached to a module of matrices. For the enumeration of conjugacy classes, we can assume that $\mathbf{G}$ is obtained from an alternating bilinear map by a variant of the classical Baer correspondence. In either case, our counting problem is naturally related to the study of rank loci within modules of matrices; apart from classical constructions, this builds upon work of O'Brien and Voll [6].

Enumerating matrices of given rank is known to be a geometrically "wild" problem [1]. This has interesting consequences for the enumeration of orbits. For example, one can construct explicit examples of $\mathbf{G} \leqslant \mathrm{U}_n$ such that the total number of orbits of $\mathbf{G}(\mathbf{F}_q)$ on $\mathbf{F}_q^n$ is a polynomial in q, but such that the number of orbits of $\mathbf{G}(\mathbf{F}_q)$ of size q^i (for a suitable fixed i) is not PORC as q ranges over primes. (For an example, combine [3, §1.7] and [8, §4].) Beyond explicit constructions, by combining [1] and [9], one can show that there exist examples of this type in which the non-PORC behaviour is arbitrarily wild in a precise sense.

The main part of the talk revolved around the enumeration of conjugacy classes by means of generating functions. Drawing upon work of du Sautoy [4], the *class-counting zeta function* of a group scheme $\mathbf{G}$ over a ring R is the Dirichlet series

$$\zeta^{\mathrm{cc}}_{\mathbf{G}}(s) = \sum_{I \triangleleft R} \mathrm{k}(\mathbf{G}(R/I))\,|R/I|^{-s},$$

where the sum extends over ideals of finite index of R and $\mathrm{k}(G)$ denotes the number of conjugacy classes of a group G. Given $\mathbf{G} \leqslant \mathrm{GL}_n$, the study of $\zeta^{\mathrm{cc}}_{\mathbf{G}\otimes\mathfrak{O}}(s)$ as $\mathfrak{O}$ ranges over compact discrete valuation rings is of particular interest. Henceforth, q denotes the residue field size of $\mathfrak{O}$. In many cases of interest, there are "geometric formulae" for $\zeta^{\mathrm{cc}}_{\mathbf{G}\otimes\mathfrak{O}}(s)$ that combine finitely many rational functions in q and q^{-s} and the numbers of $(\mathfrak{O}/\mathfrak{P})$-rational points of schemes derived from $\mathbf{G}$. Excluding small residue characteristics, such formulae have been obtained for Chevalley groups [2] and for unipotent groups [7] in characteristic zero.

As a tentative definition, by a symbolic computation of $\mathrm{k}(\mathbf{G}(\mathfrak{O}/\mathfrak{P}^m))$ for fixed $\mathbf{G} \leqslant \mathrm{U}_n$ and varying $\mathfrak{O}$ (of large residue characteristic) and m, we mean the explicit construction of a geometric formula of the aforementioned type for $\zeta^{\mathrm{cc}}_{\mathbf{G}\otimes\mathfrak{O}}(s)$. While possible in principle, this definition leads to theoretical and practical issues. For example, it seems to be unknown whether equality of two such formulae is even decidable. In practice, many examples of class-counting (and other) zeta functions of interest turn out to be *uniform* in the sense that given $\mathbf{G}$, there exists a single rational function $W(X,T) \in \mathbf{Q}(X,T)$ such that for all $\mathfrak{O}$ (subject perhaps to restrictions on its characteristic or residue characteristic), $\zeta^{\mathrm{cc}}_{\mathbf{G}\otimes\mathfrak{O}}(s) = W(q, q^{-s})$. In that case, our problem of symbolically enumerating conjugacy classes is tantamount to computing $W(X,T)$.

Zeta functions enumerating linear orbits and conjugacy classes of unipotent groups can be usefully regarded as special cases of *ask zeta functions* [7]. The latter functions are obtained by averaging over sizes of kernels within suitable parameterisations of modules of matrices. This averaging operation is related to the enumeration of orbits via a Lie-theoretic linearisation of the orbit-counting lemma. The study of ask zeta functions combines established results from p-adic integration and algebraic duality operations ("Knuth duality") [8].

The problem of computing class-counting zeta functions has a particularly satisfactory solution for *graphical group schemes*. Given a graph Γ with distinct vertices $v_1, \dots, v_n$, the associated graphical group scheme $\mathbf{G}_\Gamma$ generalises a number of constructions in the literature. In particular, for an odd prime p, the group $\mathbf{G}_\Gamma(\mathbf{F}_p)$ is the maximal quotient of class at most 2 and exponent dividing p of the right-angled Artin group $\langle x_1, \dots, x_n \mid [x_i, x_j] = 1 \text{ whenever } v_i \not\sim v_j\rangle$. Class-counting zeta functions associated with graphical group schemes turn out to be uniform in a very strong sense: given Γ, there exists $W_\Gamma(X,T) \in \mathbf{Q}(X,T)$ such that for each compact discrete valuation ring $\mathfrak{O}$ as above, $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma\otimes\mathfrak{O}}(s) = W_\Gamma(q, q^{-s})$; see [9, Cor. B]. Thanks to a constructive proof, these rational functions can be explicitly computed, at least for small graphs. They also exhibit a rich combinatorial structure, in particular for *cographs* [9, Thms C–D].

The final part of the talk contained a brief overview of some further developments. Topics discussed included Lins's work [5] on bivariate conjugacy class and representation zeta functions, steps [3, Thm E] towards understanding class-counting zeta functions of group schemes derived from free nilpotent Lie algebras, and the enumeration of conjugacy classes of graphical groups over finite fields [10].

References

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