Computing zeta functions of groups, algebras, and modules

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Over the past decades, zeta functions associated with algebraic counting problems have received considerable attention. In particular, following the seminal paper [3] of Grunewald, Segal, and Smith, the theory of subobject zeta functions evolved into a distinct branch of asymptotic algebra.

While the initial focus in the area was on the enumeration of subgroups of finitely generated nilpotent groups, it was already observed in [3] that the Mal'cev correspondence all but reduces this problem to the enumeration of subalgebras of associated nilpotent Lie algebras. More formally, let R be $\mathbf{Z}$ or the ring $\mathbf{Z}_p$ of p-adic integers. Then, given a possibly non-associative R-algebra L whose underlying R-module is free of finite rank d, we define the **subalgebra zeta function** of L to be $\zeta_L(s) = \sum_{n=1}^{\infty} a_n(L) n^{-s}$, where $a_n(L)$ denotes the number of R-subalgebras of L of additive index n and s is a complex variable. It is easy to see that if L is a $\mathbf{Z}$-algebra, then we obtain the Euler product factorisation $\zeta_L(s) = \prod_p \zeta_{L\otimes\mathbf{Z}_p}(s)$, where p ranges over all primes. A deep result from [3], derived using non-constructive model-theoretic techniques, asserts that each **local zeta function** $\zeta_{L\otimes\mathbf{Z}_p}(s)$ is a rational function in p^{-s}. In another key paper in the area, du Sautoy and Grunewald [2] showed that, excluding finitely many exceptional primes, the functions $\zeta_{L\otimes\mathbf{Z}_p}(s)$ can all be expressed in terms of a single formula. Specifically, they showed that there are $\mathbf{Q}$-varieties $V_1, \dots, V_r$ and rational functions $W_1, \dots, W_r \in \mathbf{Q}(X, Y)$ such that, for almost all primes p,

$$\zeta_{L\otimes\mathbf{Z}_p}(s) = \sum_{i=1}^{r} \#\bar{V}_i(\mathbf{F}_p)\cdot W_i(p, p^{-s}), \qquad (\star)$$

where $\bar{\cdot}$ denotes "reduction modulo p". While their proof is constructive, it is usually impractical due to its reliance on resolution of singularities.

This talk was devoted to describing a practical method [5] for computing a formula $(\star)$ in favourable situations. This method combines techniques from a number of areas. In particular, it relies on

- the formalism for expressing local subobject zeta functions in terms of p-adic integrals from [2, 3],
- results from singularity theory and toric geometry due to Khovanskii [4] and others,
- algorithms of Barvinok and others from computational convex geometry (see, in particular, [1]), and
- ideas from the theory of Gröbner bases.

In practice, we can frequently do much better than merely producing a formula ($\star$). Namely, for many examples of interest, the $\zeta_{L\otimes \mathbf{Z}_p}(s)$ are "uniform" in the sense that there exists a single rational function $W \in \mathbf{Q}(X,Y)$ such that $\zeta_{L\otimes\mathbf{Z}_p}(s) = W(p,p^{-s})$ for almost all primes p; our goal is then to find W. Among other things, this involves symbolically counting rational points on certain types of varieties.

As an application, we discussed the computation of the subalgebra zeta function of $\mathfrak{gl}_2(\mathbf{Z}_p)$ for $p \gg 0$. We also presented the author's "semi-simplification conjecture" [6, Conj. E] which asserts that given a rational unital matrix algebra, the behaviour of its associated generic local submodule zeta functions at zero only depends on the action of the largest semi-simple quotient of the algebra.

References

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