Habilitation Thesis

Zeta Functions of Groups, Algebras, and Modules

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Contents

Introduction v

0. A framework for computing zeta functions of groups, algebras, and modules . 1

1. Computing topological zeta functions of groups, algebras, and modules, I . . . 27

2. Computing topological zeta functions of groups, algebras, and modules, II . . 63

3. Topological representation zeta functions of unipotent groups 103

4. Computing local zeta functions of groups, algebras, and modules 131

5. Enumerating submodules invariant under an endomorphism 173

6. Stability results for local zeta functions of groups and related structures . . . 201

Introduction

The six research articles [10–15] (Chapters 1–6) collected in this thesis contain contributions to the theory of zeta functions of groups, algebras, and modules. To give an example of the types of zeta functions considered, let A be a possibly non-associative $\mathbf{Z}$-algebra whose underlying $\mathbf{Z}$-module is finitely generated, e.g. a matrix Lie algebra over $\mathbf{Z}$ or an order in a finite-dimensional associative algebra over $\mathbf{Q}$. The **(global) subalgebra zeta function** of A is the Dirichlet series

$$\zeta_{\mathsf{A}}^{\leqslant}(s) = \sum_{n=1}^{\infty} a_n^{\leqslant}(\mathsf{A}) \cdot n^{-s},$$

where $a_n^{\leqslant}(\mathsf{A})$ denotes the number of subalgebras of A of additive index n. Following a landmark paper by Grunewald, Segal, and Smith [7], a theory of these and related zeta functions (arising e.g. from the enumeration of subgroups, ideals, submodules, or representations of suitable algebraic structures) developed rapidly. Moreover, as the theory evolved, deep and unexpected connections between algebra, number theory, algebraic geometry, model theory, and combinatorics have been revealed.

It is an elementary observation (essentially the Chinese remainder theorem) that the subalgebra zeta function $\zeta_{\mathsf{A}}^{\leqslant}(s)$ from above admits an Euler product factorisation

$$\zeta_{\mathsf{A}}^{\leqslant}(s) = \prod_p \zeta_{\mathsf{A}\otimes\mathbf{Z}_p}^{\leqslant}(s), \tag{$*$}$$

where p ranges over primes, $\mathbf{Z}_p$ denotes the ring of p-adic integers, and each **local subalgebra zeta function** $\zeta_{\mathsf{A}\otimes\mathbf{Z}_p}^{\leqslant}(s)$ enumerates the subalgebras of finite index of the $\mathbf{Z}_p$-algebra $\mathsf{A} \otimes \mathbf{Z}_p$—or, equivalently, the subalgebras of p-power index of A.

The factorisation $(*)$ provides the main motivation for encoding the numbers $a_n^{\leqslant}(\mathsf{A})$ from above in a Dirichlet series rather than, say, an ordinary generating function. Note that the subalgebra zeta function of $\mathbf{Z}$ is the Riemann zeta function $\zeta(s)$ whose associated factorisation $(*)$ is given by Euler's formula $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.

Despite deep results of du Sautoy and Grunewald [5], the analytic properties of global zeta functions such as $\zeta_{\mathsf{A}}^{\leqslant}(s)$ often remain elusive. In fact, in contrast to the example of the Riemann zeta function from above, even the study of the local zeta functions $\zeta_{\mathsf{A}\otimes\mathbf{Z}_p}^{\leqslant}(s)$ associated with A frequently poses formidable challenges.

The majority of the work recorded in this thesis is devoted to local and so-called **topological zeta functions**. The latter zeta functions were introduced by Denef and Loeser [3] as certain limits "$p \to 1$" of local ones. While topological zeta functions contain less information than their local relatives, they have frequently been found to be more amenable to both theoretical investigations and explicit computations.

Results I: methods for explicit computations. Building heavily upon each other, the four articles [10–13] (Chapters 1–4) connect the theory of zeta functions of algebraic structures with another branch of mathematics: computational algebra. These articles are devoted to the development of practical methods for explicitly computing various types of local and topological zeta functions in "fortunate cases" related to geometric genericity conditions. In the author's opinion, there is presently little reason to expect the existence of a method for computing the types of zeta functions considered here which is both general and practical. The crucial feature of the author's work is therefore practicality which is demonstrated by an implementation in the publicly available software package Zeta [17].

An overview of [10–13], including a more detailed introduction to the area, is given in the survey article [16] (Chapter 0); the latter article is intended as a chapter in a forthcoming book on the DFG Priority Programme "Algorithmic and Experimental Methods in Algebra, Geometry and Number Theory" (SPP 1489) which funded the author's research from 2013 to 2016.

The remainder of this introduction is devoted to summarising those of the author's contributions which are contained in this thesis but not among the computational results discussed in [16].

Results II: convex-geometric formulae for local and topological zeta functions. Apart from providing the theoretical foundation of the author's computations in [11–13], the main result of [10] is of independent interest: [10, Thm 4.10] gives an explicit convex-geometric formula for a large class of p-adic integrals under non-degeneracy assumptions on defining polynomials with respect to associated Newton polytopes. The author's formula vastly generalises previous results of Denef and Hoornaert [2] and Bories [1] in the realm of Igusa's local zeta function. Similarly, the "topological counterpart" [10, Thm 6.7] of [10, Thm 4.10] generalises a result of Denef and Loeser [3, Thm 5.3(i)]. In addition to [2, 3], the proofs of [10, Thms 4.10 and 6.7] make essential use of work of Khovanskii [8, 9] and others in toric geometry.

Results III: topological representation zeta functions. As indicated above, topological zeta functions (associated with polynomials, at first) were introduced by Denef and Loeser [3] as limits "$p \to 1$" of local zeta functions; a rigorous account of this process is a rather subtle matter. They later [4] found another interpretation of topological zeta functions of polynomials in the context of motivic integration, a point of view subsequently used by du Sautoy and Loeser [6, §8] to introduce topological subalgebra zeta functions.

Apart from providing techniques for their explicit computation (and numerous applications of these), [12] (Chapter 3) includes the first rigorous definition of topological representation zeta functions of unipotent groups. Moreover, [12] contains a number of theoretical results on these zeta functions, including, in particular, a proof that they always have degree zero [12, Cor. 4.7]. As explained in [10, §8.1], the degree of a topological zeta function reflects curious properties of its local relatives which, to the author's knowledge, previously escaped attention.

Results IV: submodules invariant under a matrix. The study of the so-called submodule zeta functions featuring (among other types of zeta functions) in [10,11,13] goes back to work of L. Solomon [20]. The article [14] (Chapter 5) constitutes a thorough analysis of a special class of submodule zeta functions. Given an $n \times n$ matrix A with entries in the ring of S-integers $\mathfrak{o}_S$ of a number field k (where S is a finite set of places of k containing the Archimedean ones), consider the zeta function

$$\zeta_{A,\mathfrak{o}_S}(s) = \sum_{m=1}^{\infty} a_m(A, \mathfrak{o}_S) m^{-s},$$

where $a_m(A, \mathfrak{o}_S)$ denotes the number of A-invariant submodules of $\mathfrak{o}_S^n$ of additive index m. We regard such zeta functions as arithmetic analogues of the varieties of subspaces invariant under a given matrix as studied e.g. by Shayman [19].

Disregarding a finite number of exceptional Euler factors, [14, Thm A] expresses $\zeta_{A,\mathfrak{o}_S}(s)$ in an explicit fashion (depending on the rational canonical form of A over k) as a product of translates of Dedekind zeta functions. Particular consequences include the fact that, unlike general submodule zeta functions, $\zeta_{A,\mathfrak{o}_S}(s)$ always admits meromorphic continuation to the complex plane. Moreover, the abscissa of convergence of $\zeta_{A,\mathfrak{o}_S}(s)$ (known to be a positive rational number by deep and general results of du Sautoy and Grunewald [5]) turns out to be a natural number.

In addition to further results on analytic properties of the zeta functions $\zeta_{A,\mathfrak{o}_S}(s)$, [14] also contains applications to other types of zeta functions. In particular, [14, Prop. 6.2] shows that global ideal zeta functions associated with non-abelian nilpotent Lie algebras of maximal class have abscissa of convergence 2; this constitutes one of very few instances where abscissae of convergence of subobject zeta functions have been determined without explicitly computing the latter (a usually infeasible task).

Finally, by [14, Thm 4.4], the ideal zeta function of the power series ring $\mathbf{Z}[\![X]\!]$ is

$$\zeta^{\triangleleft}_{\mathbf{Z}[\![X]\!]}(s) = \prod_{i=1}^{\infty} \zeta(is - i + 1),$$

where $\zeta(s)$ again denotes the Riemann zeta function. Segal [18] has shown that for suitable Dedekind domains R, the ideal zeta function of $R[X]$ is $\zeta^{\triangleleft}_{R[X]}(s) = \prod_{i=1}^{\infty} \zeta^{\triangleleft}_{R}(is - i)$. To the author's knowledge, these two formulae constitute the only cases of known ideal zeta functions of 2-dimensional rings.

While logically independent of the computational work in [10–13], calculations using Zeta were instrumental in finding the statements of the main results of [14].

Results V: stability. The shortest article [15] (Chapter 6) in this thesis is devoted to the interplay of two natural operations in the context of the local zeta functions from above: variation of the prime and local base extensions. Instead of reproducing the main result [15, Thm 3.2] of [15] and its implications for zeta functions of algebraic structures, we give an example of a typical application.

First, we recall that the **(twist) representation zeta function** $\widetilde{\zeta^{\mathrm{irr}}_{G}}(s)$ of a finitely generated nilpotent pro-p group G (where p is a prime) is the Dirichlet series

$$\widetilde{\zeta^{\mathrm{irr}}_{G}}(s) = \sum_{n=1}^{\infty} \tilde{r}_n(G) \cdot n^{-s},$$

where $\tilde{r}_n(G)$ denotes the (finite) number of continuous irreducible n-dimensional complex representations of G, counted up to equivalence and tensoring with continuous 1-dimensional representations.

Let $\mathbf{G}$ and $\mathbf{H}$ be unipotent algebraic groups over $\mathbf{Q}$. By choosing faithful linear representations of $\mathbf{G}$ and $\mathbf{H}$, we obtain group schemes G and H (over $\mathbf{Z}$) such that G and H become isomorphic to $\mathbf{G}$ and $\mathbf{H}$, respectively, after extension of scalars. Since $\mathbf{G}$ and $\mathbf{H}$ are unipotent, $\mathsf{G}(\mathbf{Z}_p)$ and $\mathsf{H}(\mathbf{Z}_p)$ are finitely generated nilpotent pro-p groups for almost every prime p (i.e. for all but finitely many p). The main result of [15] implies that the representation zeta functions of these groups are "rigid" in the following sense. Suppose that $\widetilde{\zeta^{\mathrm{irr}}_{\mathsf{G}(\mathbf{Z}_p)}}(s) = \widetilde{\zeta^{\mathrm{irr}}_{\mathsf{H}(\mathbf{Z}_p)}}(s)$ for all p in a set of primes of density 1. Then for almost all primes p and all finite extensions K of the field $\mathbf{Q}_p$ of p-adic numbers, $\widetilde{\zeta^{\mathrm{irr}}_{\mathsf{G}(\mathfrak{O}_K)}}(s) = \widetilde{\zeta^{\mathrm{irr}}_{\mathsf{H}(\mathfrak{O}_K)}}(s)$, where $\mathfrak{O}_K$ denotes the valuation ring of K.

The proofs of such results in [15] combine formulae obtained using the p-adic integration machinery frequently used in the area, Grothendieck's trace formula for the number of rational points on varieties over finite fields, and Chebotarev's density theorem. Further applications include consequences for the computation of local and topological zeta functions and the interpretation of local functional equations such as those established by Voll [21].

Bibliography

[1] B. Bories, *Igusa's p-adic local zeta function associated to a polynomial mapping and a polynomial integration measure*, Manuscripta Math. **138** (2012), no. 3-4, 395–417.

[2] J. Denef and K. Hoornaert, *Newton polyhedra and Igusa's local zeta function*, J. Number Theory **89** (2001), no. 1, 31–64.

[3] J. Denef and F. Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc. **5** (1992), no. 4, 705–720.

[4] ———, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), no. 3, 505–537.

[5] M. P. F. du Sautoy and F. J. Grunewald, *Analytic properties of zeta functions and subgroup growth*, Ann. of Math. (2) **152** (2000), no. 3, 793–833.

[6] M. P. F. du Sautoy and F. Loeser, *Motivic zeta functions of infinite-dimensional Lie algebras*, Selecta Math. (N.S.) **10** (2004), no. 2, 253–303.

[7] F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), no. 1, 185–223.

[8] A. G. Khovanskii, *Newton polyhedra, and toroidal varieties*, Funkcional. Anal. i Priložen. **11** (1977), no. 4, 56–64, 96.

[9] ———, *Newton polyhedra, and the genus of complete intersections*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 1, 51–61.

[10] T. Rossmann, *Computing topological zeta functions of groups, algebras, and modules, I*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 5, 1099–1134.

[11] ———, *Computing topological zeta functions of groups, algebras, and modules, II*, J. Algebra **444** (2015), 567–605.

[12] ———, *Topological representation zeta functions of unipotent groups*, J. Algebra **448** (2016), 210–237.

[13] ———, *Computing local zeta functions of groups, algebras, and modules (preprint)* (2016). `http://www.math.uni-bielefeld.de/~rossmann/files/padzeta.pdf`.

[14] ———, *Enumerating submodules invariant under an endomorphism*, Math. Ann. (2016), 1–27. doi:10.1007/s00208-016-1499-6.

[15] ———, *Stability results for local zeta functions of groups and related structures (preprint)* (2015). `http://www.math.uni-bielefeld.de/~rossmann/files/stability.pdf`.

[16] ———, *A framework for computing zeta functions of groups, algebras, and modules (preprint)* (2016). `http://www.math.uni-bielefeld.de/~rossmann/files/1489.pdf`.

[17] ———, Zeta*, version 0.3.1*, 2016. See `http://www.math.uni-bielefeld.de/~rossmann/Zeta/`.

[18] D. Segal, *Ideals of finite index in a polynomial ring*, Quart. J. Math. Oxford Ser. (2) **48** (1997), no. 189, 83–92.

[19] M. A. Shayman, *On the variety of invariant subspaces of a finite-dimensional linear operator*, Trans. Amer. Math. Soc. **274** (1982), no. 2, 721–747.

[20] L. Solomon, *Zeta functions and integral representation theory*, Advances in Math. **26** (1977), no. 3, 306–326.

[21] C. Voll, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), no. 2, 1181–1218.