

On the enumeration of orbits of unipotent groups over finite fields

“Where the wild things are”

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We show that the enumeration of linear orbits and conjugacy classes of \mathbf{Z} -defined unipotent groups over finite fields is “wild” in the following sense: given an arbitrary scheme Y of finite type over \mathbf{Z} and integer $n \geq 1$, the numbers $\#Y(\mathbf{F}_q) \bmod q^n$ can be expressed, uniformly in q , in terms of the numbers of linear orbits (or numbers of conjugacy classes) of finitely many \mathbf{Z} -defined unipotent groups over \mathbf{F}_q and finitely many Laurent polynomials in q .

1 Introduction

Linear orbits and conjugacy classes. This article is devoted to what might be called the “symbolic enumeration” of linear orbits and conjugacy classes of families of matrix groups. Let $k(G)$ denote the number of conjugacy classes (or **class number**) of a group G . For a (commutative, unital) ring R , let $\mathrm{GL}_n(R)$ denote the group of invertible $n \times n$ matrices over R , and let $\mathrm{U}_n(R) \leq \mathrm{GL}_n(R)$ be the subgroup of upper unitriangular matrices.

It is well-known that $k(\mathrm{GL}_n(\mathbf{F}_q))$ is given by a polynomial in q ; see [27, Exercise 1.190]. Enumerating linear orbits (i.e. orbits of linear groups on their natural modules) often turns out to be an easier task than enumerating conjugacy classes. For example, while proving that $k(\mathrm{GL}_n(\mathbf{F}_q))$ depends polynomially on q is elementary, it is a complete triviality that $\#(\mathbf{F}_q^n/\mathrm{GL}_n(\mathbf{F}_q)) = 2$ for $n \geq 1$. Whether $k(\mathrm{U}_n(\mathbf{F}_q))$ depends polynomially on q or not is precisely the subject of “Higman’s conjecture”; see e.g. [13, 18, 29]. Again, linear orbits are much more easily enumerated: it is an easy exercise to show that $\#(\mathbf{F}_q^n/\mathrm{U}_n(\mathbf{F}_q)) = nq - n + 1$.

Linear algebraic groups. The functors GL_n and U_n are (represented by) group schemes over \mathbf{Z} . In this article, by a **linear algebraic group** G over \mathbf{Q} , we mean a closed subgroup scheme of $\mathrm{GL}_n \otimes \mathbf{Q}$ for some $n \geq 1$. Equivalently, we may think of such a G as being encoded by a Zariski-closed subgroup of $\mathrm{GL}_n(\mathbf{C})$ whose vanishing ideal is generated by polynomials with rational coefficients. Given $G \leq \mathrm{GL}_n \otimes \mathbf{Q}$, by clearing denominators, for each ring R , we obtain a group $G(R) \leq \mathrm{GL}_n(R)$. (More formally, letting \mathcal{G} denote the scheme-theoretic closure of G in GL_n , a group scheme over \mathbf{Z} , we abuse notation and write $G(R)$ for $\mathcal{G}(R)$.)

It is well known that every unipotent algebraic group over \mathbf{Q} is isomorphic to a closed subgroup scheme of $\mathrm{U}_n \otimes \mathbf{Q}$ for some n ; see e.g. [6, Ch. IV, §2, Prop. 2.5]. This article is a contribution to the following.

Question 1.1. Given a unipotent algebraic group $G \leq \mathrm{U}_n \otimes \mathbf{Q}$ over \mathbf{Q} , how do the numbers $k(G(\mathbf{F}_q))$ and $\#(\mathbf{F}_q^n/G(\mathbf{F}_q))$ vary with the prime power q ?

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Here, the underlying prime p of $q = p^f$ is allowed to vary, as is the exponent $f \geq 1$. By the **symbolic enumeration** of the orbits in Question 1.1, we mean the (arguably vaguely defined) task of providing meaningful descriptions of the counting functions of q given there. For example, the tamest case occurs when these counting functions are given by polynomials in q , as they are e.g. for $G = \mathrm{U}_3 \otimes \mathbf{Q}$ (the Heisenberg group).

Remark 1.2. Assuming that we are willing to discard finitely many (potentially) exceptional characteristics in Question 1.1, it turns out that it suffices (in a precise sense) to consider class numbers and numbers of linear orbits over \mathbf{F}_p for *primes* p , rather than over all \mathbf{F}_q for *prime powers* q ; see Lemma 2.4 and the comments that follow it.

Geometric bounds. For an upper bound on the complexity of the counting functions of q in Question 1.1, the orbit-counting lemma from elementary group theory shows that understanding either one of $k(G(\mathbf{F}_q))$ and $\#(\mathbf{F}_q^n/G(\mathbf{F}_q))$ is at most as hard as enumerating \mathbf{F}_q -rational points on schemes of finite type over \mathbf{Z} ; cf. Corollary 2.3.

Main result: wildness mod q^n . Given that the study and enumeration of rational points on schemes is an extraordinarily challenging and deep task, one might wonder whether the upper bound on the complexity of the functions of q in Question 1.1 provided by the preceding paragraph truly reflects the actual difficulty of enumerating orbits of unipotent groups. Our main result (Theorem A) shows that it does—after a fashion.

As explained in [24, §2.4] (and partially recalled in §3.4), each alternating bilinear map $*$: $\mathbf{Z}^d \times \mathbf{Z}^d \rightarrow \mathbf{Z}^e$ gives rise to an associated *Baer group scheme* G_* . This is a unipotent group scheme of class at most 2 with underlying scheme \mathbf{A}^{d+e} . The group $G_*(\mathbf{Z})$ is torsion-free, nilpotent of class at most 2, and it has Hirsch length $d + e$.

Theorem A. *Let Y be a scheme of finite type over \mathbf{Z} . Let $n \geq 1$. Then there are finitely many commutative group schemes M_1, \dots, M_r with $M_i \leq \mathrm{U}_{d_i}$, Baer group schemes G_1, \dots, G_r , and univariate Laurent polynomials $f_1(X), \dots, f_r(X), g_1(X), \dots, g_r(X) \in \mathbf{Z}[X^{\pm 1}]$ such that for each prime power q , the numbers $F(q) := \sum_{i=1}^r f_i(q) k(G_i(\mathbf{F}_q))$ and $G(q) := \sum_{i=1}^r g_i(q) \#(\mathbf{F}_q^{d_i}/M_i(\mathbf{F}_q))$ are integers which satisfy*

$$\#Y(\mathbf{F}_q) \equiv F(q) \equiv G(q) \pmod{q^n}. \quad (1.1)$$

Our proof of Theorem A will combine a deep result due to Belkale and Brosnan [1] and machinery developed in [21, 23, 24]. The author does not know if the congruences in (1.1) can be elevated to honest equations (perhaps at the cost of considering a larger ring of rational functions than $\mathbf{Z}[X^{\pm 1}]$ for the coefficients $f_i(X)$ and $g_i(X)$). Theorem A is however strong enough to establish that the class numbers and numbers of linear orbits in Theorem A are as geometrically “wild” as possible. Indeed, given any scheme Y of finite type over \mathbf{Z} , there exists $n \geq 1$ such that $\#Y(\mathbf{F}_q) < q^n$ for all prime powers q . Equation (1.1) therefore allows us to recover the precise number $\#Y(\mathbf{F}_q)$ from expressions involving Laurent polynomials in q as well as the class numbers $k(G_i(\mathbf{F}_q))$ and numbers of linear orbits $\#(\mathbf{F}_q^{d_i}/M_i(\mathbf{F}_q))$.

Remark 1.3. Generalising work of Boston and Isaacs [3, Thm 3.2], O’Brien and Voll [17, Thm 4.1] showed that every sufficiently generic Pfaffian hypersurface H over \mathbf{Z} gives rise to a unipotent group scheme G such that $k(G(\mathbf{F}_q))$ depends in an explicit form on $\#H(\mathbf{F}_q)$ (for odd q). All explicit examples of unipotent group schemes G for which $k(G(\mathbf{F}_q))$ depends “geometrically” on q known to the author are essentially of this form.

Remark 1.4. While this will play no role in our proof of Theorem A, we note that said theorem reduces to the case that Y is affine. That is, we may assume that $Y = \mathrm{Spec}(\mathbf{Z}[X_1, \dots, X_\ell]/(f_1, \dots, f_r))$, where $f_1, \dots, f_r \in \mathbf{Z}[X_1, \dots, X_\ell]$. In that case, $\#Y(\mathbf{F}_q) = \#\{x \in \mathbf{F}_q^\ell : f_1(x) = \dots = f_r(x) = 0\}$.

To reduce Theorem A to the affine case, we may proceed as in the proof of [5, Prop. 3.5.3]. Namely, suppose that Theorem A holds for affine Y . By subtraction, it then also holds in the quasi-affine case. For general Y , consider a finite cover $Y = \bigcup_{i \in I} Y_i$ in which each Y_i is an open affine subscheme of Y . Writing $Y_J = \bigcap_{j \in J} Y_j$ for $\emptyset \neq J \subset I$, each Y_J is then quasi-affine and the claim follows from the inclusion-exclusion principle.

Remark 1.5. In our proof of Theorem A, the group schemes M_1, \dots, M_r and G_1, \dots, G_r will generally depend on n . However, we will see in Corollary 4.11 that the number r can be chosen to be independent of n .

Related counting problems. There are numerous natural enumerative problems in which, for each prime p or prime power q , we seek to count objects up to a suitable equivalence relation. For example, given a group scheme G acting on a scheme X , we may consider the numbers $|X(\mathbf{F}_q)/G(\mathbf{F}_q)|$; this is the theme of the present paper. Other counting problems in the literature are of a less (obviously) geometric origin. In particular, let $f(n, p)$ denote the number of isomorphism classes of groups of order p^n . The study of $f(n, p)$ as a function of p (for fixed n) is precisely the subject of Higman's famous and open **PORC conjecture** [14]: it predicts that for fixed n , the function $p \mapsto f(n, p)$ is polynomial on residue classes (relative to a suitable modulus). This conjecture has been verified for $n \leq 7$; see [16] as well as [2] and the references therein.

Suspensions that Higman's PORC conjecture might be false have been expressed, see [9], in particular. In [9], this was based on the authors's prediction that certain non-PORC functions are "extremely unlikely" to add up to PORC functions. (We note that while [24] is concerned with very different enumerative problems, [24, Cor. 1.2] and [1, Thm 0.5] together show that thanks to cancellations, natural examples of highly non-PORC functions can indeed add up to polynomials.) Du Sautoy [7, Cor. 1.9] showed that for fixed n , the number $f(n, p)$ is expressible in terms of the numbers of \mathbf{F}_p -rational points of schemes over \mathbf{Z} . One might well suspect that this (rather than the PORC conjecture) captures the true general shape of the functions $p \mapsto f(n, p)$.

While Theorem A does not constitute a contribution towards resolving Higman's PORC conjecture either way, it does suggest that natural group-theoretic counting problems can indeed be as wild as arithmetic geometry allows; cf. [8, Problem 4.3].

Low nilpotency class dominates. Theorem A allows us to approximate the numbers $\#Y(\mathbf{F}_q)$ in terms of quantities derived from unipotent group schemes of class at most 2. As we will now indicate, it is perhaps no coincidence that higher nilpotency class plays no role here. Indeed, let $G \leq U_n \otimes \mathbf{Q}$ be a unipotent algebraic group over \mathbf{Q} . Then the following proposition shows that (up to an increase of dimensions and the exclusion of finitely many exceptional characteristics), the enumeration of the linear orbits and conjugacy classes of $G(\mathbf{F}_q)$ reduces to the case that G has nilpotency class at most 2.

Proposition B. *Let $G \leq U_n \otimes \mathbf{Q}$ be a unipotent algebraic group of dimension d over \mathbf{Q} .*

- (i) *There exist a commutative algebraic group $M \leq U_{2n} \otimes \mathbf{Q}$ of dimension d over \mathbf{Q} and an integer $u \geq 1$ such that $\#(\mathbf{F}_q^n/G(\mathbf{F}_q)) = q^{-n} \#(\mathbf{F}_q^{2n}/M(\mathbf{F}_q))$ for all prime powers q with $\gcd(q, u) = 1$.*
- (ii) (Cf. [24, Prop. 2.5]). *There exist a unipotent algebraic group H of class at most 2 and dimension $3d$ over \mathbf{Q} and an integer $v \geq 1$ such that $k(G(\mathbf{F}_q)) = q^{-d} k(H(\mathbf{F}_q))$ for all prime powers q with $\gcd(q, v) = 1$.*

Proposition B(ii) follows from [24, Prop. 2.5], a result pertaining to class-counting zeta functions of unipotent group schemes. One motivation for writing this article was to illustrate how some of the machinery surrounding "ask zeta functions" developed in [21, 23, 24] can be applied to enumerative problems involving groups over finite fields.

Overview. In §2, we construct rings which we will use to investigate counting problems depending on prime powers q . The counting problems that we consider give rise to elements of rings \mathfrak{Q} and \mathfrak{Q}^\flat . The former of these rings embeds as a dense subring into a certain topological ring \mathfrak{Q}_q . In §3, we mostly collect known material related to average sizes of kernels and we explain how this relates to the enumeration of orbits of unipotent groups. This, in particular, gives rise to chains of submodules of \mathfrak{Q} and \mathfrak{Q}_q . Finally, in §4, we prove Theorem A. In fact, we deduce the latter theorem from Theorem C which asserts that several of the aforementioned submodules of \mathfrak{Q}_q all have the same closure.

Notation

The symbol “ \subset ” indicates not necessarily proper inclusion. Rings are assumed to be commutative and unital. Schemes (including group schemes) are assumed to be separated and of finite type over their bases, the default of the latter being \mathbf{Z} . Maps and groups usually act on the right. Throughout, \mathcal{P} denotes the set of all rational primes and $\mathcal{D} = \{p^e : p \in \mathcal{P}, e \geq 1\}$ the set of prime powers.

2 Rings from counting problems

This section takes inspiration from the combinatorial motives of Belkale and Brosnan [1] and Huber’s affinoid rings [15].

2.1 Constructing \mathfrak{Z}^* and \mathfrak{Q}^*

Define \mathfrak{Z} to be the subring of the product $\prod_{\mathcal{D}} \mathbf{Z}$ consisting of elements $x \in \prod_{\mathcal{D}} \mathbf{Z}$ that are polynomially bounded in the sense that given x , there exists a constant $d \geq 0$ with $|x_q| \leq q^d$ for all $q \in \mathcal{D}$. For example, if X is a scheme, then $\#X := (|X(\mathbf{F}_q)|)_{q \in \mathcal{D}}$ belongs to \mathfrak{Z} . (Recall that schemes are assumed to be of finite type.) Write $\mathfrak{q} = \# \mathbf{A}^1 = (q)_{q \in \mathcal{D}} \in \mathfrak{Z}$. If F is any set-valued functor on the category of rings which is either a sub- or quotient functor of $X(\cdot)$ for a scheme X , then $\#F := (|F(\mathbf{F}_q)|)_{q \in \mathcal{D}} \in \mathfrak{Z}$. For example, if G is a group scheme acting on a scheme X , then the naive orbit space functor X/G given by $R \rightsquigarrow X(R)/G(R)$ yields an element $\#(X/G) = (|X(\mathbf{F}_q)/G(\mathbf{F}_q)|)_{q \in \mathcal{D}} \in \mathfrak{Z}$. As a special case, for any group scheme G , we write $\mathbf{k}(G) = (\mathbf{k}(G(\mathbf{F}_q)))_{q \in \mathcal{D}} \in \mathfrak{Z}$.

Discarding small primes. We will often ignore finitely many potentially exceptional characteristics. To that end, consider the ideal \mathfrak{J} of \mathfrak{Z} given by

$$\mathfrak{J} = \left\{ x \in \mathfrak{Z} : \exists N \geq 1 \forall q \in \mathcal{D} (\gcd(q, N) = 1 \rightarrow x_q = 0) \right\}.$$

Define $\mathfrak{Z}^\flat = \mathfrak{Z}/\mathfrak{J}$. Hence, $x, y \in \mathfrak{Z}$ are identified in \mathfrak{Z}^\flat if and only if $x_{p^e} = y_{p^e}$ for almost all (i.e. for all but finitely many) primes $p \in \mathcal{P}$ and all $e \geq 1$. By abuse of notation, we often use the same symbol for an element of \mathfrak{Z} and for its image in \mathfrak{Z}^\flat . Henceforth, a superscript “ $*$ ” refers to either “ \flat ” or the absence of a decoration, the convention being that “ $*$ ” has the same meaning within a given statement. For instance, $\mathfrak{q} \in \mathfrak{Z}^*$ is algebraically independent over \mathbf{Z} and \mathfrak{Z}^* embeds into the localisation $\mathfrak{Q}^* := \mathfrak{Z}^*[q^{-1}] = \mathfrak{Z}^* \otimes_{\mathbf{Z}[q]} \mathbf{Z}[q^{\pm 1}]$. Note that we may view \mathfrak{Q} as a subring of $\prod_{\mathcal{D}} \mathbf{Q}$.

Lemma 2.1. $\bigcap_{n=1}^{\infty} \mathfrak{q}^n \mathfrak{Z}^* = 0$.

Proof. First note that $\bigcap_{n=1}^{\infty} \mathfrak{q}^n \mathfrak{Z} \subset \bigcap_{n=1}^{\infty} \prod_{q \in \mathcal{D}} q^n \mathbf{Z} = 0$. Let $x \in \mathfrak{Z}$ and suppose that $x + \mathfrak{J} \in \bigcap_{n=1}^{\infty} \mathfrak{q}^n \mathfrak{Z}^\flat$. Since $x \in \mathfrak{Z}$, there exists $d \geq 1$ such that $|x_q| \leq q^d$ for all $q \in \mathcal{D}$. As $x + \mathfrak{J} \in \mathfrak{q}^{d+1} \mathfrak{Z}^\flat$, there exists $N \geq 1$ such that $q^{d+1} \mid x_q$ for all $q \in \mathcal{D}$ with $\gcd(q, N) = 1$. We conclude that $x_q = 0$ whenever $\gcd(q, N) = 1$ whence $x \in \mathfrak{J}$. Thus, $\bigcap_{n=1}^{\infty} \mathfrak{q}^n \mathfrak{Z}^\flat = 0$. \blacklozenge

2.2 Integral forms and the enumeration of orbits

Let X_0 be a scheme over \mathbf{Q} . Choose an integral form X of X_0 . That is, choose a scheme X with $X \otimes \mathbf{Q} \approx X_0$ over \mathbf{Q} . (The existence of such an X follows by *spreading out*; see e.g. [19, §3.2].) While $\#X \in \mathbf{Z}$ depends on the choice of X , the image of $\#X$ in \mathbf{Z}^b only depends on X_0 . (Indeed, another spreading out argument shows that given integral forms X and X' of X_0 , we obtain an isomorphism $X \otimes \mathbf{Z}[1/N] \approx X' \otimes \mathbf{Z}[1/N]$ over $\mathbf{Z}[1/N]$ for some $N \geq 1$. We then have $|X(\mathbf{F}_q)| = |X'(\mathbf{F}_q)|$ for all $q \in \mathcal{D}$ with $\gcd(q, N) = 1$ whence $\#X = \#X'$ in \mathbf{Z}^b .) We let $\#X_0$ denote the image of $\#X$ in \mathbf{Z}^b .

Let $G_0 \leq \mathrm{GL}_n \otimes \mathbf{Q}$ be a linear algebraic group over \mathbf{Q} . The scheme-theoretic closure G of G_0 in GL_n is an integral form of G_0 as a group scheme; see e.g. [11, §1]. Define $\#(\mathbf{A}_\mathbf{Q}^n/G_0)$ (resp. $\mathbf{k}(G_0)$) to be the image of $\#(\mathbf{A}^n/G)$ (resp. $\mathbf{k}(G)$) in \mathbf{Z}^b . Note that $\mathbf{k}(G_0)$ does not depend on the choice of the embedding of G_0 into $\mathrm{GL}_n \otimes \mathbf{Q}$.

2.3 Counting orbits: geometric perspectives

Let G be a group scheme acting on a scheme X . While the naive orbit space functor X/G is hardly ever (represented by) a scheme, the point-counting function $q \mapsto |X(\mathbf{F}_q)/G(\mathbf{F}_q)|$ *does* behave somewhat similarly to the one attached to a scheme. To see this, let $X \diamond G$ be the closed subscheme of $X \times G$ representing $R \rightsquigarrow \{(x, g) \in X(R) \times G(R) : xg = x\}$.

Lemma 2.2. $|X(R)/G(R)| = |(X \diamond G)(R)| \cdot |G(R)|^{-1}$ for each finite ring R .

Proof. This follows immediately from the orbit-counting lemma. ◆

Corollary 2.3. Let G be a group scheme acting on a scheme X . Suppose that $G \otimes \mathbf{Q}$ is a unipotent algebraic group of dimension d over \mathbf{Q} . Then $\#(X/G) = \#(X \diamond G) \cdot q^{-d}$ in \mathbf{Q}^b .

Proof. $G \otimes \mathbf{Q} \approx \mathbf{A}_\mathbf{Q}^d$ as schemes over \mathbf{Q} ; cf. [6, Ch. IV, §2, Prop. 4.1]. ◆

When our objective is to enumerate orbits of group scheme actions on the level of rational points over finite fields, assuming that we are willing to ignore finitely many exceptional characteristics, then it often turns out to be enough to consider prime fields. More formally, inspired by properties of the point-counting functions $p \mapsto |Y(\mathbf{F}_p)|$ attached to a scheme Y as described in [26, Ch. 1], we observe the following.

Lemma 2.4. For each $i = 1, \dots, r$, let G_i be a group scheme acting on a scheme Y_i . Let $f_1(X), \dots, f_r(X) \in \mathbf{Q}[X^{\pm 1}]$. For $q \in \mathcal{D}$, define $F(q) = \sum_{i=1}^r |Y_i(\mathbf{F}_q)/G_i(\mathbf{F}_q)| \cdot f_i(q)$. Suppose that $F(p) = 0$ for all $p \in P$, where $P \subset \mathcal{P}$ has natural density 1. Then there exists $N \geq 1$ such that $F(q) = 0$ for all $q \in \mathcal{D}$ with $\gcd(q, N) = 1$. That is, $(F(q))_{q \in \mathcal{D}}$ vanishes in \mathbf{Q}^b .

Proof. Let $Z_i = (Y_i \diamond G_i) \times \prod_{j \neq i} G_j$ and $\tilde{F}(q) = \sum_{i=1}^r |Z_i(\mathbf{F}_q)| f_i(q)$. By Lemma 2.2 and since $G_i(\mathbf{F}_q) \neq \emptyset$, for each $q \in \mathcal{D}$, we have $F(q) = 0$ if and only if $\tilde{F}(q) = 0$. It thus suffices to establish the analogous claim for \tilde{F} . This follows by combining Chebotarev's density theorem and Grothendieck's trace formula; cf. [22, Thm 2.3]. ◆

We record two particular consequences:

- (i) Let G_i act on Y_i for $i = 1, 2$. Suppose that $|Y_1(\mathbf{F}_p)/G_1(\mathbf{F}_p)| = |Y_2(\mathbf{F}_p)/G_2(\mathbf{F}_p)|$ for almost all primes p . Then in fact $|Y_1(\mathbf{F}_q)/G_1(\mathbf{F}_q)| = |Y_2(\mathbf{F}_q)/G_2(\mathbf{F}_q)|$ for all prime powers q except perhaps for a finite number of exceptional characteristics. (For trivial actions, this is [26, Thm 1.3].)

- (ii) Let G act on Y and suppose that $|Y(\mathbf{F}_p)/G(\mathbf{F}_p)|$ is given by a (rational) polynomial in p for almost all primes p . Then $|Y(\mathbf{F}_q)/G(\mathbf{F}_q)|$ too depends polynomially on q except perhaps for a finite number of exceptional characteristics.

Remark 2.5. For related geometric perspectives on the enumeration of conjugacy classes, see [20, §2] and [10, §2].

2.4 Completing \mathbb{Z} and \mathbb{Q}

Let $\mathbb{Z}_q = \varprojlim \mathbb{Z}/q^n \mathbb{Z}$ be the q -adic completion of \mathbb{Z} . As $\bigcap_{n=1}^{\infty} q^n \mathbb{Z} = 0$, we may regard \mathbb{Z} as a (dense) subring of \mathbb{Z}_q . Since q is a regular element of \mathbb{Z}_q , the latter ring embeds into $\mathbb{Q}_q := \mathbb{Z}_q[q^{-1}]$. We regard \mathbb{Q} as a subring of \mathbb{Q}_q . We topologise \mathbb{Q}_q via the identification $\mathbb{Q}_q = \varinjlim (\mathbb{Z}_q \xrightarrow{q} \mathbb{Z}_q \xrightarrow{q} \cdots)$. This turns \mathbb{Q}_q into a topological ring which contains \mathbb{Z}_q as an open and closed subring. The subspace topology of \mathbb{Z}_q within \mathbb{Q}_q coincides with our original q -adic topology on \mathbb{Z}_q .

As usual, for $p \in \mathcal{P}$, let \mathbf{Z}_p and \mathbf{Q}_p denote the ring of p -adic integers and field of p -adic numbers, respectively¹. For $e \geq 1$, we write $\mathbf{Z}_{p^e} = \mathbf{Z}_p$ and $\mathbf{Q}_{p^e} = \mathbf{Q}_p$; note that $\mathbf{Z}_q = \varprojlim \mathbf{Z}/q^n \mathbf{Z}$ for $q \in \mathcal{D}$. Let $\mathbb{Z}_n = \prod_{q \in \mathcal{D}} \mathbf{Z}/q^n \mathbf{Z}$, regarded as a discrete ring. As $\mathbb{Z}_n \approx \mathbb{Z}/q^n$ naturally, we may identify $\mathbb{Z}_q = \varprojlim \mathbb{Z}_n$ as topological rings. Since products commute with inverse limits (both being categorical limits), the natural embeddings $\mathbb{Z} \hookrightarrow \prod_{\mathcal{D}} \mathbf{Z}$ and $\mathbb{Q} \hookrightarrow \prod_{\mathcal{D}} \mathbf{Q}$ extend to a continuous isomorphism of abstract rings (not a homeomorphism!) $\mathbb{Z}_q \rightarrow \prod_{q \in \mathcal{D}} \mathbf{Z}_q$ and a continuous ring monomorphism $\mathbb{Q}_q \rightarrow \prod_{q \in \mathcal{D}} \mathbf{Q}_q$, $x \mapsto (x_q)_{q \in \mathcal{D}}$, respectively.

Proposition 2.6. $\mathbb{Q} \cap \mathbb{Z}_q = \mathbb{Z}$.

Proof. Let $x \in \mathbb{Q} \cap \mathbb{Z}_q$. Then there exists $m \geq 0$ and $y \in \mathbb{Z}$ such that $x_q = y_q/q^m$ for all $q \in \mathcal{D}$. As $x_q \in \mathbf{Z}_q$, we conclude that $x_q \in \mathbf{Z}$ for all $q \in \mathcal{D}$. The converse is clear. \blacklozenge

We may view \mathbb{Z}_q as $\prod_{q \in \mathcal{D}} \mathbf{Z}_q$ endowed with a suitable topology of uniform convergence:

Lemma 2.7. Let $U \subset \mathbb{Q}_q$ and $x \in \mathbb{Z}_q$. Then $x \in \bar{U}$ if and only if for each $n \geq 1$, there exists $u \in U \cap \mathbb{Z}_q$ such that $x_q \equiv u_q \pmod{q^n}$ for all $q \in \mathcal{D}$.

Proof. Since \mathbb{Z}_q is both open and closed in \mathbb{Q}_q , we have $\bar{U} \cap \mathbb{Z}_q = \overline{U \cap \mathbb{Z}_q}$; cf. [4, Ch. I, §1, no. 6, Prop. 5]. Let V_n denote the image of $U \cap \mathbb{Z}_q$ in the discrete space \mathbb{Z}_n . The claim follows since $\overline{U \cap \mathbb{Z}_q} = \varprojlim V_n$; see [4, Ch. I, §4, no. 4, Cor.]. \blacklozenge

It is easy to see that \mathbf{Z} is a closed and discrete subring of \mathbb{Z}_q (hence of \mathbb{Q}_q). Moreover, the subspace topology of $\mathbf{Z}[q]$ in \mathbb{Z}_q (and in \mathbb{Q}_q) is q -adic, and we may identify $\overline{\mathbf{Z}[q]} = \mathbf{Z}[[q]] = \varprojlim \mathbf{Z}[q]/q^n$ within \mathbb{Z}_q .

Remark 2.8. (i) While we do not need this here, the same steps as above allow us to define \mathbb{Z}_q^b as the q -adic completion of \mathbb{Z}^b , and then construct $\mathbb{Q}_q^b = \mathbb{Z}_q^b[q^{-1}]$. One can show that the natural map $\mathbb{Q}_q \rightarrow \mathbb{Q}_q^b$ (induced by the quotient map $\mathbb{Z} \rightarrow \mathbb{Z}^b$) is continuous, open, and surjective. A reader might wonder why we chose to define \mathbb{Z} as a particular subring of $\prod_{\mathcal{D}} \mathbf{Z}$ instead of simply defining $\mathbb{Z} = \prod_{\mathcal{D}} \mathbf{Z}$. (Both choices give rise to the same q -adic completion.) Our definition of \mathbb{Z} ensures that \mathbb{Z}^b embeds into \mathbb{Z}_q^b . (ii) The pair $(\mathbb{Q}_q, \mathbb{Z}_q)$ is an example of a (non-Noetherian) *affinoid ring*; see [15, §1.1] or [25, §§2.1–2.2] for background. In particular, our construction above is a special case of a much more general completion procedure.

¹Our construction of \mathbb{Z}_q and \mathbb{Q}_q is an imitation of the usual construction of \mathbf{Z}_p and \mathbf{Q}_p , respectively.

3 Average sizes of kernels and the enumeration of orbits

In §§3.1–3.4, we mostly collect known material from [21, 23, 24] pertaining to the enumeration of linear orbits and conjugacy classes of unipotent groups. We then derive Proposition B in §3.5. Finally, in §3.6, we deduce equations and inclusions between various submodules of \mathfrak{Q}^* ; these relations will play a crucial role in our proof of Theorem A.

3.1 Module representations

For more on the following, see [23, §§2, 4, 7]. Let R be a ring. A **module representation** over R is a homomorphism $M \xrightarrow{\theta} \text{Hom}(V, W)$, where M , V , and W are R -modules. Equivalently, θ is determined by the bilinear map $*$: $V \times M \rightarrow W$ given by $x * a = x(a\theta)$ for $a \in M$ and $x \in V$. (Recall that we usually write maps on the right.) Given a (commutative, associative, unital) R -algebra S , we obtain a module representation $M \otimes_R S \xrightarrow{\theta^S} \text{Hom}(V \otimes_R S, W \otimes_R S)$ over S derived from θ by base change. For $m \geq 0$, let ${}^m\theta$ denote the module representation $M \rightarrow \text{Hom}(V^{\oplus m}, W^{\oplus m})$ over R with $a({}^m\theta) = (a\theta)^{\oplus m}$ for $a \in M$; we identify ${}^m({}^n\theta) = {}^{mn}\theta$.

Let $(\cdot)^* = \text{Hom}(\cdot, R)$. A module representation θ over R as above gives rise to further module representations over R , referred to as the *Knuth duals* of θ in [23, §4]. Among these, we will need the module representation $W^* \xrightarrow{\theta^\bullet} \text{Hom}(V, M^*)$ given by

$$x(\psi\theta^\bullet) = (M \rightarrow R, a \mapsto (x(a\theta))\psi). \quad (\psi \in W^*, x \in V)$$

Two module representations $M \xrightarrow{\theta} \text{Hom}(V, W)$ and $\tilde{M} \xrightarrow{\tilde{\theta}} \text{Hom}(\tilde{V}, \tilde{W})$ over R with bilinear maps $*$ and $\tilde{*}$ are **isotopic** (written $\theta \approx \tilde{\theta}$) if there exists a triple of module isomorphisms $(M \xrightarrow{\nu} \tilde{M}, V \xrightarrow{\phi} \tilde{V}, W \xrightarrow{\psi} \tilde{W})$ such that $(x * a)\psi = (x\phi)\tilde{*}(a\nu)$ for all $a \in M$ and $x \in V$. The **alternating hull** $\Lambda(\theta)$ of a module representation $M \xrightarrow{\theta} \text{Hom}(V, W)$ over R is the module representation

$$V \oplus M \rightarrow \text{Hom}(V \oplus M, W), \quad (x, a) \mapsto ((x', a') \mapsto x'(a\theta) - x(a'\theta)).$$

Let $M \xrightarrow{\theta} \text{Hom}(V, W)$ be a module representation over R . We say that θ is **finite free** if each of M , V , and W is free of finite rank. Writing ℓ , d , and e for these ranks, each choice of bases of all the modules involved gives rise to an isotopy from $R^\ell \rightarrow \text{M}_{d \times e}(R) = \text{Hom}(R^d, R^e)$ to θ . If θ is finite free, then so is θ^\bullet and, moreover, $\theta \approx \theta^{\bullet\bullet}$. Finally, we say that θ is **immersive** if it is injective, finite free, and if $\text{Coker}(\theta)$ is a free R -module. (This notion did not feature in the work cited above.) In that case, after choosing bases as above and identifying $\text{M}_{d \times e}(R) = R^{de}$, the module representation θ gives rise to a closed immersion $\mathbf{A}_R^\ell \rightarrow \mathbf{A}_R^{de}$. If θ is immersive, then so are ${}^m\theta$ and $\Lambda(\theta^\bullet)^\bullet$; this is clear in the former case and follows from [23, Remark 7.10] and $\theta \approx \theta^{\bullet\bullet}$ in the latter case.

If no ring is specified, then all module representations are understood to be over \mathbf{Z} . Note that a module representation over \mathbf{Z} (or any PID) is immersive if and only if it is injective, finite free, and has torsion-free cokernel.

3.2 Average sizes of kernels

Let $M \xrightarrow{\theta} \text{Hom}(V, W)$ be a module representation over a ring R . Suppose that M and V are finite as sets. As in [23, §1], the **average size** of the **kernel** of the linear maps parameterised by θ is $\text{ask}(\theta) = \frac{1}{|M|} \sum_{a \in M} |\text{Ker}(a\theta)|$. Let θ be a finite free module representation over \mathbf{Z} . Define $\underline{\text{ask}}(\theta) = \left(\text{ask}(\theta^{\mathbf{F}_q}) \right)_{q \in \mathcal{D}} \in \mathfrak{Q}$. This quantity only depends on the isotopy class of θ , and the image of $\underline{\text{ask}}(\theta)$ in \mathfrak{Q}^\flat only depends on the isotopy class of $\theta^{\mathbf{Q}}$ over \mathbf{Q} .

Lemma 3.1 (Cf. [23, Eqn (5.1)]). *Let $M \xrightarrow{\theta} \text{Hom}(V, W)$ be a finite free module representation. Let C be the scheme representing the functor $R \rightsquigarrow \{(x, a) \in (V \otimes R) \times (M \otimes R) : x(a\theta^R) = 0\}$. Then $\underline{\text{ask}}(\theta) = q^{-\text{rk}(M)} \cdot \#C$ in \mathcal{Q} .*

When investigating $\underline{\text{ask}}(\theta)$ in \mathcal{Q}^b , it suffices to consider immersive module representations.

Lemma 3.2. *Let $M \xrightarrow{\theta} \text{Hom}(V, W)$ be a finite free module representation over \mathbf{Z} . Then there exists an immersive module representation $\tilde{M} \xrightarrow{\tilde{\theta}} \text{Hom}(V, W)$ over \mathbf{Z} such that $\underline{\text{ask}}(m\theta) = \underline{\text{ask}}(m\tilde{\theta})$ in \mathcal{Q}^b for all $m \geq 0$.*

Proof. Let $M/\text{Ker}(\theta) \xrightarrow{\bar{\theta}} \text{Hom}(V, W)$ be induced by θ . As $M/\text{Ker}(\theta)$ is torsion-free, $\bar{\theta}$ is finite free. Clearly, $\underline{\text{ask}}(m\theta) = \underline{\text{ask}}(m\bar{\theta})$ (in \mathcal{Q}) for all $m \geq 0$ so we may assume that θ is the inclusion of a submodule M into $\text{Hom}(V, W)$. Let \tilde{M}/M be the torsion submodule of $\text{Hom}(V, W)/M$. Let $\tilde{\theta}$ be the inclusion of \tilde{M} into $\text{Hom}(V, W)$; note that $\tilde{\theta}$ is immersive. Then \tilde{M}/M has finite cardinality, N say, and $\tilde{M}/M \otimes \mathbf{F}_q = 0$ for $q \in \mathcal{D}$ with $\gcd(q, N) = 1$. For such q , the natural map $M \otimes \mathbf{F}_q \rightarrow \tilde{M} \otimes \mathbf{F}_q$ is then onto and the claim follows. \blacklozenge

3.3 Counting linear orbits of unipotent groups

From module representations to abelian groups. Let $M \xrightarrow{\theta} M_{d \times e}(\mathbf{Z})$ be a finite free module representation. Define M_θ to be the group scheme representing $R \rightsquigarrow M \otimes R$ (additive group). Let $*$ be the bilinear map associated with θ as in §3.1. The action of M on $\mathbf{Z}^{d+e} = \mathbf{Z}^d \oplus \mathbf{Z}^e$ given by $(x, y)a = (x, x * a + y)$ naturally extends to an action of M_θ on \mathbf{A}^{d+e} . The second proof of [21, Lem. 2.1] shows that

$$\#(\mathbf{A}^{d+e}/M_\theta) = q^e \cdot \underline{\text{ask}}(\theta) \quad (3.1)$$

in \mathcal{Q} . (We can argue as in the proof of Lemma 3.2 to reduce to the case that θ is injective as in the setting of [21].) Since every finite free module representation is isotopic to one of the form $M \rightarrow M_{d \times e}(\mathbf{Z})$, up to a factor of the form q^e , the quantities $\underline{\text{ask}}(\theta)$ attached to finite free module representations θ enumerate orbits associated with linear actions of group schemes. If θ is immersive, then we may view M_θ as a closed (and commutative) subgroup scheme of U_{d+e} . Using Lemma 3.2, we conclude the following.

Corollary 3.3. *Let θ be a finite free module representation. (i) If θ is immersive, then there exist $m, n \geq 0$ and a closed commutative subgroup scheme $G \leq U_n$ with $\#(\mathbf{A}^n/G) = q^m \cdot \underline{\text{ask}}(\theta)$ in \mathcal{Q} . (ii) In any case, there exist $m, n \geq 0$ and a commutative algebraic subgroup $G_0 \leq U_n \otimes \mathbf{Q}$ with $\#(\mathbf{A}_{\mathbf{Q}}^n/G_0) = q^m \cdot \underline{\text{ask}}(\theta)$ in \mathcal{Q}^b .* \blacklozenge

Hence, up to a factor of the form q^e , the quantities $\underline{\text{ask}}(\theta)$ enumerate linear orbits. In the unipotent case, we obtain the following converse.

From unipotent groups to module representations. Let $G_0 \leq U_n \otimes \mathbf{Q}$ be a unipotent algebraic group over \mathbf{Q} . Let $\mathfrak{n}_n(R)$ denote the Lie algebra of strictly upper triangular $n \times n$ matrices over a ring R . We may regard the Lie algebra \mathfrak{g}_0 of G_0 as a subalgebra of $\mathfrak{n}_n(\mathbf{Q})$. Let $\mathfrak{g} = \mathfrak{g}_0 \cap \mathfrak{n}_n(\mathbf{Z})$, and let ι denote the inclusion $\mathfrak{g} \hookrightarrow M_n(\mathbf{Z})$. This is an immersive module representation.

Proposition 3.4 ([21, Proof of Propn 8.13]). $\#(\mathbf{A}_{\mathbf{Q}}^n/G_0) = \underline{\text{ask}}(\iota)$ in \mathcal{Q}^b .

3.4 Counting conjugacy classes of unipotent groups

From module representations to groups I: Heisenberg-like groups. Let $\theta: M \rightarrow \text{Hom}(V, W)$ be a module representation with corresponding bilinear map $*: V \times M \rightarrow W$. We may endow the

set $M \oplus V \oplus W$ with a group structure by formal matrix multiplication within the set $\begin{bmatrix} 1 & V & W \\ 0 & 1 & M \\ 0 & 0 & 1 \end{bmatrix}$; the required product $V \times M \rightarrow W$ is taken to be $*$. Equivalently, this group is obtained as the semidirect product of M acting on $V \oplus W$, where M acts via $(v, w)a = (v, v * a + w)$ ($a \in M$, $v \in V$, $w \in W$) (cf. §3.3). This construction of a group from θ (via $*$) is classical; for context and variations, see [30, §9.1].

By base change, the above construction of a group from θ gives rise to a group functor H_θ . If θ is finite free, then H_θ is a group scheme and $H_\theta \otimes \mathbf{Q}$ is a unipotent algebraic group of class at most 2. If $*$: $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ is the usual multiplication, then we recover the Heisenberg group scheme U_3 .

Proposition 3.5. *Let $M \xrightarrow{\theta} \text{Hom}(V, W)$ be a finite free module representation. Then $\underline{k}(H_\theta) = q^{\text{rk}(M) - \text{rk}(V) + \text{rk}(W)} \cdot \underline{\text{ask}}(^2(\theta^\bullet))$ in \mathcal{Q} .*

Proof. Combine [23, Lem. 7.7], [23, Thm 7.9], and [23, Lem. 3.2]. \blacklozenge

We note that $^2(\theta^\bullet)$ and $(^2\theta)^\bullet$ are generally not isotopic; cf. [23, Propn 4.14(i)].

From module representations to groups II: Baer group schemes. Given a module representation $M \xrightarrow{\theta} \text{Hom}(M, W)$ with corresponding bilinear map $*$: $M \times M \rightarrow W$, we say that θ is **alternating** if $a * a = 0$ for all $a \in M$. The alternating hull (see §3.1) of any module representation is alternating. Suppose that θ is alternating and finite free. As explained in [24, §2.4] (which draws upon [28, §2.4.1]), we then obtain an associated *Baer group scheme* G_θ . For each ring R , we may identify $G_\theta(R) = (M \otimes R) \oplus (W \otimes R)$ as sets. With this identification, group commutators in $G_\theta(R)$ are characterised by the identity $[(a, y), (a', y')] = (0, a *_R a')$ for $a, a' \in M \otimes R$ and $y, y' \in W \otimes R$; here $*_R$ denotes the bilinear map $(M \otimes R) \times (M \otimes R) \rightarrow W \otimes R$ induced by $*$ via base change.

Proposition 3.6 (Cf. [24, Prop. 2.6]). *Let $M \xrightarrow{\theta} \text{Hom}(M, W)$ be an alternating finite free module representation. Then $\underline{k}(G_\theta) = q^{\text{rk}(W)} \cdot \underline{\text{ask}}(\theta)$ in \mathcal{Q} .*

From unipotent groups to alternating module representations. Let G_0 be a unipotent algebraic group over \mathbf{Q} . Let \mathfrak{g} be a \mathbf{Z} -form of the Lie algebra of G_0 such that \mathfrak{g} is free of finite rank as a \mathbf{Z} -module. Let $\text{ad}_{\mathfrak{g}}: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$, $a \mapsto (x \mapsto [x, a])$ be the (right) adjoint representation of \mathfrak{g} ; note that this is an alternating finite free module representation.

Proposition 3.7 (Cf. [23, Propn 6.4]). $\underline{k}(G_0) = \underline{\text{ask}}(\text{ad}_{\mathfrak{g}})$ in \mathcal{Q}^b .

One may view the preceding proposition as a consequence of [17, Thm B]; see [23].

3.5 Reductions to low nilpotency class: proof of Proposition B

As we will now explain, when $G_0 \leq U_n \otimes \mathbf{Q}$ is a unipotent algebraic group, then the study of the \mathcal{Q}^b -valued invariants $\#(\mathbf{A}_{\mathbf{Q}}^n/G_0)$ and $\underline{k}(G_0)$ from §2 reduces to the case that G_0 has nilpotency class at most 2. (In the case of $\#(\mathbf{A}_{\mathbf{Q}}^n/G_0)$, we can even reduce to the commutative case.) While these reductions will come as no surprise to experts in the area, the author feels that they deserve to be recorded in the following form.

Lemma 3.8 (= Proposition B(i)). *Let $G_0 \leq U_n \otimes \mathbf{Q}$ be a unipotent algebraic group over \mathbf{Q} . Then there exists a commutative algebraic group $H_0 \leq U_{2n} \otimes \mathbf{Q}$ of the same dimension as G_0 such that $\#(\mathbf{A}_{\mathbf{Q}}^n/G_0) = q^{-n} \cdot \#(\mathbf{A}_{\mathbf{Q}}^{2n}/H_0)$ in \mathcal{Q}^b .*

Proof. Given G_0 , define ι as in the setting of Proposition 3.4. Then $\#(\mathbf{A}_{\mathbf{Q}}^n/G_0) = \underline{\text{ask}}(\iota)$ in \mathcal{Q}^b . Recall that ι is immersive. Using §3.3 with ι in place of θ , construct a commutative group scheme $M_\iota \leq U_{2n}$. By equation (3.1), $\#(\mathbf{A}_{\mathbf{Q}}^{2n}/M_\iota) = q^n \underline{\text{ask}}(\iota)$ in \mathcal{Q}^b . We may thus take $H_0 = M_\iota \otimes \mathbf{Q}$. \blacklozenge

Lemma 3.9 (= Proposition B(ii); cf. [24, Prop. 2.5]). *Let G_0 be a unipotent algebraic group of dimension d over \mathbf{Q} . Then there exists a unipotent algebraic group H_0 of class at most 2 and dimension $3d$ such that $\underline{k}(G_0) = q^{-d} \underline{k}(H_0)$ in \mathfrak{Q}^b .*

Proof. Given G_0 , choose \mathfrak{g} as in the setting of Proposition 3.7. Then $\underline{k}(G_0) = \underline{\text{ask}}(\text{ad}_{\mathfrak{g}})$ in \mathfrak{Q}^b . Let $H_0 = G_{\text{ad}_{\mathfrak{g}}} \otimes \mathbf{Q}$. By Proposition 3.6, $\underline{k}(H_0) = q^d \underline{\text{ask}}(\text{ad}_{\mathfrak{g}}) = q^d \underline{k}(G_0)$ in \mathfrak{Q}^b . \blacklozenge

3.6 Equations & inclusions: from \mathfrak{A}^* to \mathfrak{X}^*

Recall that \mathfrak{Q}^* denotes either \mathfrak{Q} or \mathfrak{Q}^b . Define the following $\mathbf{Z}[q^{\pm 1}]$ -submodules of \mathfrak{Q}^* :

$$\begin{aligned} {}^m\mathfrak{A}^* &:= \langle \underline{\text{ask}}({}^m\theta) : \theta \text{ is a finite free module representation (over } \mathbf{Z}) \rangle_{\mathbf{Z}[q^{\pm 1}]}, \\ {}^m\mathfrak{A}_{\rightarrow}^* &:= \langle \underline{\text{ask}}({}^m\theta) : \theta \text{ is an immersive module representation (over } \mathbf{Z}) \rangle_{\mathbf{Z}[q^{\pm 1}]}, \\ {}^{-}\mathfrak{A}^* &:= \langle \underline{\text{ask}}(\theta) : \theta \text{ is an alternating finite free module representation} \rangle_{\mathbf{Z}[q^{\pm 1}]}, \text{ and} \\ \mathfrak{X}^* &:= \langle \underline{\#}X : X \text{ is a scheme} \rangle_{\mathbf{Z}[q^{\pm 1}]}. \end{aligned}$$

It is easy to see that these $\mathbf{Z}[q^{\pm 1}]$ -submodules of \mathfrak{Q}^* are in fact $\mathbf{Z}[q^{\pm 1}]$ -subalgebras. (In the cases of ${}^m\mathfrak{A}^*$ and ${}^m\mathfrak{A}_{\rightarrow}^*$ this follows since for finite free module representations θ and $\tilde{\theta}$, we have $\underline{\text{ask}}({}^m\theta) \cdot \underline{\text{ask}}({}^m\tilde{\theta}) = \underline{\text{ask}}({}^m(\theta \oplus \tilde{\theta}))$; cf. [23, Lem. 3.1].) Write $\mathfrak{A}^* := {}^1\mathfrak{A}^*$ and $\mathfrak{A}_{\rightarrow}^* := {}^1\mathfrak{A}_{\rightarrow}^*$. Clearly, ${}^{mn}\mathfrak{A}^* \subset {}^m\mathfrak{A}^*$ and ${}^{mn}\mathfrak{A}_{\rightarrow}^* \subset {}^m\mathfrak{A}_{\rightarrow}^*$ for $m, n \geq 1$ and ${}^{-}\mathfrak{A}^* \subset \mathfrak{A}^*$. By Lemma 3.2, we have ${}^m\mathfrak{A}^b = {}^m\mathfrak{A}_{\rightarrow}^b$ for $m \geq 1$. The following equations and inclusions between these algebras will be of interest to us.

Proposition 3.10.

- (i) ${}^m\mathfrak{A}_{\rightarrow}^* \subset {}^m\mathfrak{A}^* \subset \mathfrak{A}^* \subset \mathfrak{X}^*$ for all $m \geq 1$.
- (ii) $\mathfrak{A}^* = \left\langle \underline{\#}(\mathbf{A}^{d+e}/M_{\theta}) : M \xrightarrow{\theta} M_{d \times e}(\mathbf{Z}) \text{ is finite free; } d, e \geq 0 \right\rangle_{\mathbf{Z}[q^{\pm 1}]}$ and $\mathfrak{A}_{\rightarrow}^* = \left\langle \underline{\#}(\mathbf{A}^{d+e}/M_{\theta}) : M \xrightarrow{\theta} M_{d \times e}(\mathbf{Z}) \text{ is immersive; } d, e \geq 0 \right\rangle_{\mathbf{Z}[q^{\pm 1}]}$.
- (iii) ${}^2\mathfrak{A}^* = \left\langle \underline{k}(H_{\theta}) : \theta \text{ is a finite free module representation} \right\rangle_{\mathbf{Z}[q^{\pm 1}]}$.
- (iv) ${}^{-}\mathfrak{A}^* = \left\langle \underline{k}(G_{\theta}) : \theta \text{ is an alternating finite free module representation} \right\rangle_{\mathbf{Z}[q^{\pm 1}]}$.
- (v) ${}^2\mathfrak{A}^* \subset {}^{-}\mathfrak{A}^*$.
- (vi) We have (with an underline added solely for emphasis)

$$\begin{aligned} \mathfrak{A}^b &= \left\langle \underline{\#}(\mathbf{A}_{\mathbf{Q}}^n/G_0) : G_0 \leq U_n \otimes \mathbf{Q} \text{ is a unipotent algebraic group, } n \geq 0 \right\rangle_{\mathbf{Z}[q^{\pm 1}]} \\ &= \left\langle \underline{\#}(\mathbf{A}_{\mathbf{Q}}^n/G_0) : G_0 \leq U_n \otimes \mathbf{Q} \text{ is a } \underline{\text{commutative}} \text{ algebraic group, } n \geq 0 \right\rangle_{\mathbf{Z}[q^{\pm 1}]}. \end{aligned}$$

- (vii) We have (with an underline added solely for emphasis)

$$\begin{aligned} {}^{-}\mathfrak{A}^b &= \left\langle \underline{k}(G_0) : G_0 \text{ is a unipotent algebraic group over } \mathbf{Q} \right\rangle_{\mathbf{Z}[q^{\pm 1}]} \\ &= \left\langle \underline{k}(G_0) : G_0 \text{ is a unipotent algebraic group } \underline{\text{of class } \leq 2} \text{ over } \mathbf{Q} \right\rangle_{\mathbf{Z}[q^{\pm 1}]}. \end{aligned}$$

Proof. (i) The first two inclusions are clear and the third follows from Lemma 3.1. (ii) This is a consequence of equation (3.1). (iii) This follows from Proposition 3.5 and the fact that for each finite free module representation θ , the dual θ^\bullet is also finite free and $\theta^{\bullet\bullet} \approx \theta$. (iv) This follows from Proposition 3.6. (v) Let $\theta: M \rightarrow \text{Hom}(V, W)$ be a finite free module representation. By [23, Thm 7.9 and Lem. 3.2], $\text{ask}(\Lambda(\theta)) = q^{\text{rk}(M) - \text{rk}(V)} \cdot \text{ask}(\Lambda(\theta^\bullet))$ in \mathcal{Q} . Hence, $\text{ask}(\Lambda(\theta)) = q^{\text{rk}(V) - \text{rk}(W)} \cdot \text{ask}(\Lambda(\theta^\bullet)) \in {}^{-\mathfrak{A}}^*$. (vi) Combine Corollary 3.3(ii), Proposition 3.4, and Lemma 3.8. (vii) Combine Proposition 3.6, Proposition 3.7, and Lemma 3.9. \blacklozenge

In particular, we obtain the following chain of equations and inclusions:

$$\begin{aligned} \mathfrak{X} \supset \mathfrak{A} &= \left\langle \#(\mathbf{A}^{d+e}/M_\theta) : M \xrightarrow{\theta} M_{d \times e}(\mathbf{Z}) \text{ is finite free} \right\rangle_{\mathbf{Z}[q^{\pm 1}]} \\ &\supset {}^{-\mathfrak{A}} = \left\langle k(G_\theta) : \theta \text{ is alternating and finite free} \right\rangle_{\mathbf{Z}[q^{\pm 1}]} \\ &\supset {}^2\mathfrak{A} \supset {}^2\mathfrak{A}_{\rightarrow}. \end{aligned} \tag{3.2}$$

While the author does not know if the inclusion ${}^2\mathfrak{A}_{\rightarrow} \subset \mathfrak{X}$ is strict, Theorem C will imply that ${}^2\mathfrak{A}_{\rightarrow}$ and \mathfrak{X} have the same closure within the completion \mathcal{Q}_q of \mathcal{Q} constructed in §2.4. Theorem A then follows easily from this; see §4.3.

Remark 3.11. In [1, §0.2], the subring CMot^+ of \mathcal{Z} additively generated by all $\#X$ as X ranges over schemes is called the ring of *effective combinatorial motives*. As we will explain and exploit in §4, our ring \mathfrak{X} lies between CMot^+ and the ring CMot of *combinatorial motives* from [1].

4 Main results: Theorems A and C

Recall the definitions of ${}^m\mathfrak{A}_{\rightarrow}$ and \mathfrak{X} from §3.6. In this section, we prove the following.

Theorem C. *Let $m \geq 1$. Then $\overline{{}^m\mathfrak{A}_{\rightarrow}} = \overline{\mathfrak{X}}$ in \mathcal{Q}_q .*

As we will see, Theorem C is essentially a stronger form of Theorem A. The role of immersive (rather than merely finite free) module representations in our arguments is arguably subtle: they allow us to view the M_i in Theorem A as closed subgroup schemes of the U_{d_i} ; see Remark 4.9.

4.1 Rank loci and limits of average sizes of kernels

Let $M \xrightarrow{\theta} \text{Hom}(V, W)$ be a finite free module representation (over \mathbf{Z}). Let M , V , and W have ranks ℓ , d , and e , respectively. By choosing a \mathbf{Z} -basis of M , we may identify $M = \mathbf{Z}^\ell$. Let D_i^θ be the subscheme of \mathbf{A}^ℓ representing the functor

$$R \rightsquigarrow \left\{ a \in M \otimes R : \bigwedge^{i+1} (a\theta^R) = 0 \right\}.$$

Remark 4.1. The D_i^θ are simply the preimages in \mathbf{A}^ℓ of the *degeneracy loci* of the collections of linear maps $\mathbf{A}^d \rightarrow \mathbf{A}^d$ parameterised by θ ; cf. [12, Ch. 14]. We can obtain an explicit description of D_i^θ by replacing θ by an isotopic copy $\mathbf{Z}^\ell \rightarrow M_{d \times e}(\mathbf{Z})$. In this setting, θ is equivalently described by a matrix of linear forms $A(X) \in M_{d \times e}(\mathbf{Z}[X_1, \dots, X_\ell])$ characterised by $a\theta = A(a)$ for $a \in \mathbf{Z}^\ell$. The scheme D_i^θ is the subscheme of \mathbf{A}^ℓ defined by the vanishing of the $(i+1) \times (i+1)$ minors of $A(X)$.

Let $V_i^\theta = D_i^\theta \setminus D_{i-1}^\theta$ for $i > 0$ and $V_0^\theta = D_0^\theta$. For our purposes, the most important property of the V_i^θ is that for each field K , we have $V_i^\theta(K) = \{a \in M \otimes K : \text{rk}(a\theta^K) = i\}$. This allows us to express average sizes of kernels derived from θ in terms of the V_i^θ .

4 Main results

Lemma 4.2 (Cf. [21, §2.1]). $\underline{\text{ask}}(^m\theta) = q^{-\ell} \sum_{i=0}^d \#V_i^\theta \cdot q^{m(d-i)}$ in \mathfrak{Q} .

Proof. Fix $q \in \mathcal{D}$. Write $\theta_q = \theta^{\mathbf{F}_q}$ and note that $(^m\theta)_q = ^m(\theta_q)$. We partition $M \otimes \mathbf{F}_q$ into the sets $Z_i := V_i^\theta(\mathbf{F}_q)$ for $i = 0, \dots, d$. For each $a \in Z_i$, since $\text{rk}(a\theta_q) = i$, we obtain $\text{rk}(a(^m\theta_q)) = mi$ and thus $|\text{Ker}(a(^m\theta_q))| = q^{m(d-i)}$. We therefore find that $\text{ask}(^m\theta_q) = q^{-\ell} \sum_{i=0}^d |Z_i| \cdot q^{m(d-i)}$. \blacklozenge

Write $V_{\max}^\theta := V_d^\theta$. The following simple observation is key: while $\underline{\text{ask}}(\theta)$ might be too coarse an invariant for us to recover any of the $\#V_i^\theta$ from it, we *can* recover $\#V_{\max}^\theta$ from the q -adic limit of the sequence $\underline{\text{ask}}(^m\theta)$ as $m \rightarrow \infty$.

Corollary 4.3. $\lim_{m \rightarrow \infty} \underline{\text{ask}}(^m\theta) = q^{-\ell} \cdot \#V_{\max}^\theta$ in \mathfrak{Q}_q .

Proof. For $i = 0, \dots, d-1$, the summand $\#V_i^\theta \cdot q^{m(d-i)}$ in Lemma 4.2 is divisible by q^m in \mathfrak{Z}_q . Therefore, each such summand tends to zero in \mathfrak{Z}_q as $m \rightarrow \infty$. \blacklozenge

Corollary 4.4. Let $m \geq 1$. The closure of $^m\mathfrak{A}$ (resp. $^m\mathfrak{A}_{\rightarrow}$) in \mathfrak{Q}_q contains $\#V_{\max}^\theta$ for each finite free (resp. immersive) module representation θ .

Proof. $\underline{\text{ask}}(^{mn}\theta) = \underline{\text{ask}}(^m(^n\theta)) \in ^m\mathfrak{A}$ and $\lim_{n \rightarrow \infty} \underline{\text{ask}}(^{mn}\theta) = q^{-\ell} \cdot \#V_{\max}^\theta$ in \mathfrak{Q}_q . \blacklozenge

4.2 Consequences of the work of Belkale and Brosnan

Let Γ be a (finite simple) graph. Let v_1, \dots, v_n be the distinct vertices of Γ . We use \sim to signify adjacency between vertices. Define a module of matrices

$$M_\Gamma = \left\{ x \in M_n(\mathbf{Z}) : x = x^\top \text{ and } x_{ij} = 0 \text{ whenever } v_i \sim v_j \right\}.$$

Note that M_Γ is a free \mathbf{Z} -module of rank $\binom{n+1}{2} - m$, where m denotes the number of edges of Γ . Let $\gamma = \gamma(\Gamma)$ denote the inclusion $M_\Gamma \hookrightarrow M_n(\mathbf{Z})$. It is easy to see that γ is an immersive module representation. The isotopy type of γ does not depend on the chosen ordering of the vertices of Γ .

The scheme V_{\max}^γ (as defined in §4.1) coincides with Z_Γ in the notation of [1, §§0.3, 3.2]. Let $R = \mathbf{Z}[q^{\pm 1}, (1 - q^n)^{-1} \ (n \geq 1)]$ and $\tilde{\mathfrak{Z}} = \mathfrak{Z} \otimes_{\mathbf{Z}[q]} R$. Each $q^n - 1$ for $n \geq 1$ is regular in \mathfrak{Z} . We may thus regard \mathfrak{Z} as a subring of $\tilde{\mathfrak{Z}}$ and $\tilde{\mathfrak{Z}}$ as a subring of $\prod_{\mathcal{D}} \mathbf{Q}$. The following is one of the main results of Belkale and Brosnan [1].

Theorem 4.5 (Cf. [1, Thm 0.5]). *Within $\tilde{\mathfrak{Z}}$, we have*

$$\langle \#X : X \text{ is a scheme} \rangle_R = \langle \#V_{\max}^\gamma : \Gamma \text{ is a graph} \rangle_R.$$

Remark 4.6. In [1], the module on the left-hand side of the preceding equation is denoted by CMot and called the ring of *combinatorial motives*.

4.3 Combining the pieces: proofs of Theorem C and A

As $(1 - q^n)^{-1} = \sum_{k=0}^{\infty} q^{kn}$ in \mathfrak{Z}_q , we may regard $\tilde{\mathfrak{Z}}$ as a subring of \mathfrak{Q}_q . Let $R_q = \mathbf{Z}[[q]][q^{-1}] \subset \mathfrak{Q}_q$. Then $\mathbf{Z}[q^{\pm 1}] \subset R \subset R_q$ and $\overline{\mathbf{Z}[q^{\pm 1}]} = \overline{R} = \overline{R_q} = R_q$ in \mathfrak{Q}_q . The following is elementary.

Lemma 4.7. Let B be a topological ring and let M be a topological B -module. Let $A \subset B$ be a dense subring. Let $U \subset M$ be an A -submodule and let V be the B -submodule of M generated by U . Then $\bar{U} = \bar{V}$. \blacklozenge

Corollary 4.8. $\tilde{\mathfrak{X}} = \overline{\langle \#V_{\max}^\gamma : \Gamma \text{ is a graph} \rangle_{\mathbf{Z}[q^{\pm 1}]}}$ within \mathfrak{Q}_q .

4 Main results

Proof. Given a subring $A \subset R_q$, define A -submodules $U_A = \langle \#X : X \text{ is a scheme} \rangle_A$ and $V_A = \langle \#V_{\max}^\gamma : \Gamma \text{ is a graph} \rangle_A$ of \mathfrak{Q}_q . By Lemma 4.7, $\overline{U_{\mathbf{Z}[q^{\pm 1}]}} = \overline{U_R} = \overline{U_{R_q}}$ and analogously for “ V ”. Theorem 4.5 shows that $U_R = V_R$ in \mathfrak{Q}_q . \blacklozenge

Proof of Theorem C. We know from Proposition 3.10 that ${}^m\mathfrak{A}_{\rightarrow} \subset \mathfrak{X}$. By Corollary 4.4, for each immersive module representation θ , we have $\#V_{\max}^\theta \in \overline{{}^m\mathfrak{A}_{\rightarrow}}$. Now apply the preceding corollary. \blacklozenge

Proof of Theorem A. Equation (3.2) and Theorem C together imply that $\mathfrak{A}_{\rightarrow}$ and ${}^-\mathfrak{A}$ are dense in $\overline{\mathfrak{X}}$ within \mathfrak{Q}_q . Using our descriptions of $\mathfrak{A}_{\rightarrow}$ and ${}^-\mathfrak{A}$ in terms of numbers of linear orbits and class numbers, respectively (see Proposition 3.10), Theorem A follows from Lemma 2.7. \blacklozenge

Remark 4.9. The restriction to immersive module representations (via $\mathfrak{A}_{\rightarrow}$) in the preceding proof is what allows us to view each M_i in Theorem A as a closed subgroup scheme of U_{d_i} . If we were to instead consider all finite free module representations (via \mathfrak{A}), then each M_i would still come with an associated homomorphism $M_i \rightarrow U_{d_i}$ (hence an action on \mathbf{A}^{d_i}), but this would not necessarily be a closed immersion. The conclusion of Theorem A would otherwise remain unaffected.

4.4 Towards an explicit Theorem A (mod [1])

In this final subsection, we briefly sketch a slightly more explicit way of deriving Theorem A from Theorem 4.5. This is obtained by unravelling our arguments from above. First, let $M \xrightarrow{\theta} \text{Hom}(V, W)$ be any finite free module representation. Write ℓ , d , and e for $\text{rk}(M)$, $\text{rk}(V)$, and $\text{rk}(W)$, respectively.

Lemma 4.10. *Let $m \geq 1$. Then:*

$$(i) \quad \underline{\text{ask}}({}^m\theta) = q^{-me} \cdot \#(\mathbf{A}^{m(d+e)} / M_{m\theta}).$$

$$(ii) \quad \underline{\text{ask}}(2m\theta) = q^{m(d-e)-\ell} \cdot \#(\mathbf{G}_{\Lambda((m\theta)\bullet)}).$$

Proof. (i) This follows from equation (3.1). (ii) We have $\underline{\text{ask}}(2\theta) = q^{d-e} \cdot \underline{\text{ask}}(\Lambda(\theta^\bullet))$ (see the proof of Proposition 3.10(v)). Hence, using Proposition 3.6, we obtain $\underline{\text{ask}}(2m\theta) = \underline{\text{ask}}(2(m\theta)) = q^{m(d-e)} \cdot \underline{\text{ask}}(\Lambda((m\theta)^\bullet)) = q^{m(d-e)-\ell} \cdot \#(\mathbf{G}_{\Lambda((m\theta)\bullet)})$. \blacklozenge

As abstract group schemes, $M_\theta \approx M_{m\theta}$ for all $m \geq 1$. Suppose that θ is immersive so that we may view $M_{m\theta}$ as a closed subgroup scheme of $\text{GL}_{m(d+e)}$. Within $\text{GL}_{m(d+e)}$, the group scheme $M_{m\theta}$ is then conjugate to

$$R \rightsquigarrow \left\{ \text{diag} \left(\begin{bmatrix} 1_d & a\theta^R \\ 0_{e \times d} & 1_e \end{bmatrix}, \dots, \begin{bmatrix} 1_d & a\theta^R \\ 0_{e \times d} & 1_e \end{bmatrix} \right) : a \in M \otimes R \right\}.$$

For each ring R , the underlying set of the Baer group scheme $\mathbf{G}_{\Lambda(\theta^\bullet)}(R)$ is $(W^* \otimes R) \times (V \otimes R) \times (M^* \otimes R)$. We may naturally regard the right factor $M^* \otimes R$ as a central subgroup of $\mathbf{G}_{\Lambda(\theta^\bullet)}(R)$. We leave it to the reader to verify that for each $m \geq 1$, the group $\mathbf{G}_{\Lambda((m\theta)\bullet)}(R)$ is the m -fold central power of $\mathbf{G}_{\Lambda(\theta^\bullet)}(R)$ in which the copies of $M^* \otimes R$ are identified.

Now, let the setting be as in Theorem A. By Theorem 4.5, there are graphs $\Gamma_1, \dots, \Gamma_r$ and rational functions $h_1(X), \dots, h_r(X) \in S := \mathbf{Z}[X^{\pm 1}; (1 - X^n)^{-1} \ (n \geq 1)]$ such that $\#Y = \sum_{i=1}^r h_i(q) \cdot \#V_{\max}^{\gamma_i}$; here, $\gamma_i: M_{\Gamma_i} \hookrightarrow M_{n_i}(\mathbf{Z})$ denotes the (immersive) module representation attached to Γ_i as in §4.2. Let Γ_i have m_i edges so that M_{Γ_i} has rank $\ell_i := \binom{n_i+1}{2} - m_i$ as a \mathbf{Z} -module. For $m \geq 1$, define

$$H_m := \sum_{i=1}^r h_i(q) \cdot q^{\ell_i} \cdot \underline{\text{ask}}({}^m\gamma_i).$$

By Corollary 4.3, $\lim_{m \rightarrow \infty} H_m = \#Y$ in \mathfrak{Q}_q . For each $m \geq 1$, using Lemma 4.10, we obtain (explicit) $f_i^m(X), g_i^m(X) \in S$ such that $H_m = \sum_{i=1}^r f_i^m(q) \cdot \#(\mathbf{A}^{2mn_i} / M_{m\gamma_i})$ and $H_{2m} = \sum_{i=1}^r g_i^m(q) \cdot \#(\mathbf{G}_{\Lambda((m\gamma_i)\bullet)})$.

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Fix $n \geq 1$. There exists $m \geq 1$ such that $H_m, H_{2m} \in \mathfrak{Z}_q$ and $\#Y \equiv H_m \equiv H_{2m} \pmod{q^n}$. Given m , the rational functions $f_i^m(X)$ (resp. $g_i^m(X)$) can be replaced by Laurent polynomials $f_i(X)$ (resp. $g_i(X)$) such that $f_i(q)$ (resp. $g_i(q)$) and $f_i^m(q)$ (resp. $g_i^m(q)$) are sufficiently close within \mathfrak{Q}_q . This yields Theorem A and in fact, it provides the following slight improvement.

Corollary 4.11. *The number r in Theorem A can be chosen to be independent of n .* ♦

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