Enumerating conjugacy classes of graphical groups over finite fields

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Each graph and choice of a commutative ring gives rise to an associated graphical group. In this article, we introduce and investigate graph polynomials that enumerate conjugacy classes of graphical groups over finite fields according to their sizes.

1 Introduction

1.1 Graphical groups

Throughout, graphs are finite, simple, and (unless otherwise indicated) contain at least one vertex. When the reference to an ambient graph is clear, we use $\sim$ to indicate the associated adjacency relation. All rings are associative, commutative, and unital.

Let $\Gamma = (V, E)$ be a graph with n vertices. The **graphical group** $\mathbf{G}_\Gamma(R)$ associated with Γ over a ring R was defined in [29, §3.4]. For a short equivalent description (see §2.3), write $V = \{v_1, \dots, v_n\}$ and let $J = \{(j,k) : 1 \leqslant j < k \leqslant n,\, v_j \sim v_k\}$. Then $\mathbf{G}_\Gamma(R)$ is generated by symbols $x_1(r), \dots, x_n(r)$ and $z_{jk}(r)$ for $(j,k) \in J$ and $r \in R$, subject to the following defining relations for $i, i' \in [n] := \{1, \dots, n\}$, $(j,k), (j',k') \in J$, and $r, r' \in R$:

(i) $x_i(r)x_i(r') = x_i(r + r')$ and $z_{jk}(r)z_{jk}(r') = z_{jk}(r + r')$. ("scalars")

(ii) $[x_j(r), x_k(r')] = z_{jk}(rr')$. ("adjacent vertices and commutators")
(Recall that $(j,k) \in J$ so that $v_j \sim v_k$.)

(iii) $[x_i(r), x_{i'}(r')] = 1$ if $v_i \not\sim v_{i'}$. ("non-adjacent vertices and commutators")

(iv) $[x_i(r), z_{jk}(r')] = [z_{j'k'}(r), z_{jk}(r')] = 1$. ("centrality of commutators")

Note that every ring map $R \to R'$ induces an evident group homomorphism $\mathbf{G}_\Gamma(R) \to \mathbf{G}_\Gamma(R')$. We will see in §2.3 that the resulting group functor $\mathbf{G}_\Gamma$ represents the **graphical group scheme** associated with Γ as defined in [29]. The isomorphism type of $\mathbf{G}_\Gamma$ does not depend on the chosen ordering of the vertices of Γ.

Example 1.1. Various instances and relatives of graphical groups appeared in the literature.

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(i) $\mathbf{G}_\Gamma(\mathbf{Z})$ is isomorphic to the maximal nilpotent quotient of class at most 2 of the right-angled Artin group $\langle x_1, \dots, x_n \mid [x_i, x_j] = 1 \text{ whenever } v_i \not\sim v_j \rangle$; see §2.1.

(ii) Let K_n denote a complete graph on n vertices. Then $\mathbf{G}_{\mathrm{K}_n}(\mathbf{Z})$ is a free nilpotent group group of rank n and class at most 2. For each odd prime p, the graphical group $\mathbf{G}_{\mathrm{K}_n}(\mathbf{F}_p)$ is a free nilpotent group of rank n, exponent dividing p, and class at most 2. (Both statements follow from Proposition 2.1 below. We note that the prime 2 does not play an exceptional role in any of our main results.)

(iii) Let P_n be a path graph on n vertices. Let $\mathrm{U}_d \leqslant \mathrm{GL}_d$ be the group scheme of upper unitriangular $d \times d$ matrices. Then for each ring R, the group $\mathbf{G}_{\mathrm{P}_n}(R)$ is the maximal quotient of class at most 2 of $\mathrm{U}_{n+1}(R)$; cf. [29, §9.4].

(iv) Let Δ_n denote an edgeless graph on n vertices. Then for each ring R, we may identify $\mathbf{G}_{\Delta_n}(R)$ and the (abelian) additive group R^n.

(v) Let p be an odd prime. Then $\mathbf{G}_\Gamma(\mathbf{F}_p)$ is isomorphic to the p-group attached to the complement of Γ via Mekler's construction [22]; cf. Proposition 2.1(ii). Li and Qiao [17] used what they dubbed the Baer-Lovász-Tutte procedure to attach a finite p-group to Γ. Their group is also isomorphic to $\mathbf{G}_\Gamma(\mathbf{F}_p)$; see §2.4.

1.2 Known results: class numbers of graphical groups

Let $\mathrm{cc}_e(G)$ denote the number of conjugacy classes of size e of a finite group G, and let $\mathrm{k}(G) = \sum_{e=1}^{\infty} \mathrm{cc}_e(G)$ be the **class number** of G. It is well known that $\mathrm{k}(\mathrm{GL}_d(\mathbf{F}_q))$ is a polynomial in q for fixed d; see [31, Ch. 1, Exercise 190]. This article is devoted to the class numbers $\mathrm{k}(\mathbf{G}_\Gamma(\mathbf{F}_q))$. We first recall known results.

Theorem 1.2 ([29, Cor. 1.3])**.** *Given a graph Γ, there exists $f_\Gamma(X) \in \mathbf{Q}[X]$ such that $\mathrm{k}(\mathbf{G}_\Gamma(\mathbf{F}_q)) = f_\Gamma(q)$ for each prime power q.*

We call $f_\Gamma(X)$ the **class-counting polynomial** of Γ. In [29], Theorem 1.2 is derived from a more general uniformity result [29, Cor. B] for class-counting zeta functions associated with graphical group schemes; see §8.1. Formulae for $f_\Gamma(X)$ when Γ has at most 5 vertices can be deduced from the tables in [29, §9]. Moreover, several families of class-counting polynomials have been previously computed in the literature.

Example 1.3. O'Brien and Voll [24, Thm 2.6] gave a formula for the number of conjugacy classes of given size of p-groups derived from free nilpotent Lie algebras via the Lazard correspondence. Using the interpretation of $\mathbf{G}_{\mathrm{K}_n}(\mathbf{F}_p)$ in Example 1.1(ii), their formula or, alternatively, work of Ito and Mann [16, §1] yields $f_{\mathrm{K}_n}(X) = X^{\binom{n-1}{2}}(X^n + X^{n-1} - 1)$.

Example 1.4. In light of Example 1.1(iii), Marjoram's enumeration [21, Thm 7] of the irreducible characters of given degree of the maximal class-2 quotients of $\mathrm{U}_d(\mathbf{F}_q)$ yields

$$f_{\mathrm{P}_n}(X) = \sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n-a}{a} X^{n-a-1}(X-1)^a + \binom{n-a-1}{a} X^{n-a-1}(X-1)^{a+1} \right). \qquad (1.1)$$

For any graph Γ, the size of each conjugacy class of $\mathbf{G}_\Gamma(\mathbf{F}_q)$ is of the form q^i; see Proposition 2.3(i). As indicated in [29, §8.5], the methods underpinning Theorem 1.2 can be used to strengthen said theorem: each $\mathrm{cc}_{q^i}(\mathbf{G}_\Gamma(\mathbf{F}_q))$ is a polynomial in q with rational coefficients. While constructive, the proof of Theorem 1.2 in [29] relies on an elaborate recursion. In particular, no explicit general formulae for the numbers $\mathrm{cc}_{q^i}(\mathbf{G}_\Gamma(\mathbf{F}_q))$ or the polynomial $f_\Gamma(X)$ have been previously recorded.

Recall that the **join** $\Gamma_1 \vee \Gamma_2$ of graphs Γ_1 and Γ_2 is obtained from their disjoint union $\Gamma_1 \oplus \Gamma_2$ by adding edges connecting each vertex of Γ_1 to each vertex of Γ_2. Further recall that a **cograph** is any graph that can be obtained from two cographs on fewer vertices by taking disjoint unions or joins, starting with an isolated vertex.

Theorem 1.5 (Cf. [29, Theorem E]). *Let Γ be a cograph. Then the coefficients of $f_\Gamma(X)$ as a polynomial in $X-1$ are non-negative integers.*

Theorems 1.2 and 1.5 are special cases of more general results pertaining to class numbers of graphical groups $\mathbf{G}_\Gamma(\mathfrak{O}/\mathfrak{P}^i)$, where $\mathfrak{O}$ is a compact discrete valuation ring with maximal ideal $\mathfrak{P}$. We will briefly discuss this topic in §8.

1.3 The graph polynomials $\mathscr{C}_\Gamma(X,Y)$ and $\mathsf{F}_\Gamma(X,Y)$

As before, let $\Gamma = (V,E)$ be a graph. Prior to stating our results, we first define what, to the author's knowledge, appears to be a new graph polynomial. For a (not necessarily proper) subset $U \subset V$, let $\Gamma[U]$ be the induced subgraph of Γ with vertex set U. Let $\mathrm{c}_\Gamma(U)$ denote the number of connected components of $\Gamma[U]$. (We allow $U = \emptyset$ in which case $\mathrm{c}_\Gamma(U) = 0$.) The **closed neighbourhood** $\mathrm{N}_\Gamma[v] \subset V$ of $v \in V$ consists of v and all vertices adjacent to it. For $U \subset V$, write $\mathrm{N}_\Gamma[U] = \bigcup_{u\in U} \mathrm{N}_\Gamma[u]$. Define

$$\mathscr{C}_\Gamma(X,Y) = \sum_{U\subset V} (X-1)^{|U|}\, Y^{|\mathrm{N}_\Gamma[U]|-\mathrm{c}_\Gamma(U)} \in \mathbf{Z}[X,Y]. \tag{1.2}$$

Remark 1.6. While $\mathscr{C}_\Gamma(X,Y)$ resembles the Tutte polynomial of a matroid on the ground set V, it is unclear to the author whether this is more than a formal similarity. Similarly, $\mathscr{C}_\Gamma(X,Y)$ is reminiscent of the subgraph polynomial [26, §3] of Γ.

Let Γ have n vertices, m edges, and c connected components. Recall that the **(matroid) rank** of Γ is $\mathrm{rk}(\Gamma) = n - c$. Define the **class-size polynomial** of Γ to be

$$\mathsf{F}_\Gamma(X,Y) = X^m \mathscr{C}_\Gamma(X, X^{-1}Y) = \sum_{U\subset V} (X-1)^{|U|} X^{m+\mathrm{c}_\Gamma(U)-|\mathrm{N}_\Gamma[U]|} Y^{|\mathrm{N}_\Gamma[U]|-\mathrm{c}_\Gamma(U)}. \tag{1.3}$$

The degree of $\mathscr{C}_\Gamma(X,Y)$ as a polynomial in Y is $\mathrm{rk}(\Gamma)$; see Proposition 6.2. As $\mathrm{rk}(\Gamma) \leqslant m$, we conclude that $\mathsf{F}_\Gamma(X,Y) \in \mathbf{Z}[X,Y]$. While $\mathscr{C}_\Gamma(X,Y)$ and $\mathsf{F}_\Gamma(X,Y)$ determine each other, $\mathscr{C}_\Gamma(X,Y)$ is often more convenient to work with and $\mathsf{F}_\Gamma(X,Y)$ turns out to be more directly related to the enumeration of conjugacy classes; see Theorem A.

Example 1.7.

(i) $\mathscr{C}_{\mathrm{K}_n}(X,Y) = (X^n-1)Y^{n-1} + 1$ and $\mathsf{F}_{\mathrm{K}_n}(X,Y) = \left(X^{\binom{n}{2}+1} - X^{\binom{n-1}{2}}\right)Y^{n-1} + X^{\binom{n}{2}}$.

(ii) $\mathscr{C}_{\Delta_n}(X,Y) = X^n = \mathsf{F}_{\Delta_n}(X,Y)$.

1.4 Main results

Let Γ be a graph. The main result of this article justifies the term "class-size polynomial".

Theorem A. $\mathsf{F}_\Gamma(q,Y) = \sum_{i=0}^{\infty} \mathrm{cc}_{q^i}(\mathbf{G}_\Gamma(\mathbf{F}_q))Y^i$ *for each prime power* q.

Note that $\mathrm{cc}_{q^i}(\mathbf{G}_\Gamma(\mathbf{F}_q)) = 0$ for all sufficiently large i so Theorem A asserts an equality of polynomials in Y.

Example 1.8. The formula in [24, Thm 2.6] referred to in Example 1.3 shows that if q is an odd prime power, then $\mathbf{G}_{\mathrm{K}_n}(\mathbf{F}_q)$ has a centre of order $q^{\binom{n}{2}}$ and precisely $(q^n-1)q^{\binom{n-1}{2}}$ non-trivial conjugacy classes, all of size q^{n-1}. These numbers agree with Example 1.7(i). Clearly, Example 1.7(ii) agrees with the fact that $\mathbf{G}_{\Delta_n}(\mathbf{F}_q) \approx \mathbf{F}_q^n$ is abelian.

Theorem A provides us with the following explicit formula for the class-counting polynomial $f_\Gamma(X)$ defined in Theorem 1.2.

Corollary B. $f_\Gamma(X) = \mathsf{F}_\Gamma(X,1) = \sum_{U\subset V} (X-1)^{|U|}\, X^{m+\mathrm{c}_\Gamma(U)-|\mathrm{N}_\Gamma[U]|}$. ♦

Note that Corollary B shows that $f_\Gamma(X)$ has integer coefficients. In the spirit of work surrounding Higman's conjecture (see §1.5) and Theorem 1.5, Theorem A implies the following refinement of the preceding observation.

Corollary C. *For each* $e \geqslant 1$*, the number of conjugacy classes of* $\mathbf{G}_\Gamma(\mathbf{F}_q)$ *of size* e *is given by a polynomial in* $q-1$ *with non-negative integer coefficients.*

Proof. Use the binomial theorem to expand powers of $X = (X-1)+1$ in (1.3). ♦

It is natural to ask whether the coefficients referred to in Corollary C enumerate meaningful combinatorial objects. Corollary 6.5 will provide a partial answer to this.

We shall not endeavour to improve substantially upon the exponential-time algorithm for computing $\mathscr{C}_\Gamma(X,Y)$ suggested by equation (1.2). Indeed, we will obtain the following.

Proposition D. *Computing* $\mathscr{C}_\Gamma(X,Y)$*, and hence also* $\mathsf{F}_\Gamma(X,Y)$*, is NP-hard.*

More precisely, we will see that knowledge of $\mathscr{C}_\Gamma(X,Y)$ allows us to read off the cardinalities of connected dominating sets of Γ. The problem of deciding whether a graph admits a connected dominating set of cardinality at most a given number is known to be NP-complete; see Theorem 6.6.

By Theorem A and Proposition D, symbolically enumerating the conjugacy classes of given size of $\mathbf{G}_\Gamma(\mathbf{F}_q)$ (as a polynomial in q) is NP-hard. The problem of measuring the difficulty of symbolically enumerating *all* conjugacy classes of $\mathbf{G}_\Gamma(\mathbf{F}_q)$ remains open.

Question 1.9. Is computing $f_\Gamma(X)$ NP-hard?

1.5 Related work: around Higman's conjecture

Recall that $\mathrm{U}_d \leqslant \mathrm{GL}_d$ denotes the group scheme of upper unitriangular $d \times d$ matrices. A famous conjecture due to G. Higman [14] predicts that $\mathrm{k}(\mathrm{U}_d(\mathbf{F}_q))$ is given by a polynomial in q for fixed d. This has been confirmed for $d \leqslant 13$ by Vera-López and Arregi [35] and for $d \leqslant 16$ by Pak and Soffer [25]. The former authors also showed that the sizes of conjugacy classes of $\mathrm{U}_d(\mathbf{F}_q)$ are of the form q^i (see [36, §3]) and that $\mathrm{cc}_{q^i}(\mathrm{U}_d(\mathbf{F}_q))$ is a polynomial in $q-1$ with non-negative integer coefficients for $i \leqslant d-3$ (see [34]). Many authors studied variants of Higman's conjecture for unipotent groups derived from various types of algebraic groups; see e.g. [10].

While logically independent of the work described here, Higman's conjecture (and the body of research surrounding it) certainly provided motivation for topics considered and results obtained in this article (e.g. Corollary C).

1.6 Open problems: enumerating characters of graphical groups

Let $\mathrm{Irr}(G)$ denote the set of (ordinary) irreducible characters of a finite group G. It is well known that $\mathrm{k}(G) = |\mathrm{Irr}(G)|$ (see e.g. [15, V, §5]), and the enumeration of irreducible characters of a group (according to their degrees) has often been studied as a "dual" of the enumeration of conjugacy classes (according to their sizes); see e.g. [18, 24, 28]. For odd q, [24, Thm B] implies that the degree $\chi(1)$ of each irreducible character χ of a graphical group $\mathbf{G}_\Gamma(\mathbf{F}_q)$ is of the form q^i.

Question 1.10. Let Γ be a graph and let $i \geqslant 0$ be an integer. How does

$$\mathrm{ch}(\Gamma, i; q) := \#\left\{\chi \in \mathrm{Irr}(\mathbf{G}_\Gamma(\mathbf{F}_q)) : \chi(1) = q^i\right\}$$

depend on the prime power q?

It is known that $\mathrm{ch}(\Gamma, i; q)$ is a polynomial in q for $\Gamma = \Delta_n$ (trivially), $\Gamma = \mathrm{P}_n$ (by [21, Thm 7]), and $\Gamma = \mathrm{K}_n$ (for odd q; by [24, Prop. 2.4].

Let $v_1, \dots, v_n$ be the distinct vertices of a graph Γ. Let Y consist of algebraically independent variables Y_{ij} (over $\mathbf{Z}$) indexed by pairs (i, j) with $1 \leqslant i < j \leqslant n$ and $v_i \sim v_j$. Let $B_\Gamma(Y)$ be the antisymmetric $n \times n$ matrix whose (i, j) entry for $i < j$ is equal to Y_{ij} if $v_i \sim v_j$ and zero otherwise. (That is, $B_\Gamma(Y)$ is a generic antisymmetric matrix with support constraints defined by Γ as in [29].) Let m be the number of edges of Γ. Using an arbitrary ordering, relabel our variables as $Y = (Y_1, \dots, Y_m)$. Then [24, Thm B] shows that for odd q, up to a factor given by an explicit power of q (depending on Γ and i), $\mathrm{ch}(\Gamma, i; q)$ coincides with $\#\left\{y \in \mathbf{F}_q^m : \mathrm{rk}_{\mathbf{F}_q}(B_\Gamma(y)) = 2i\right\}$. In our proof of Theorem A (see §4), the number of conjugacy classes of $\mathbf{G}_\Gamma(\mathbf{F}_q)$ of given size is similarly expressed in terms of the number of specialisations of given rank of a matrix of linear forms. In that setting, the latter enumeration can be carried out explicitly using algebraic and graph-theoretic arguments.

It is unclear to the author whether such a line of attack could be used to answer Question 1.10. Work of Belkale and Brosnan [2, Thm 0.5] on rank counts for generic *symmetric* (rather than antisymmetric) matrices with support constraints leads the author to suspect that the functions of q considered in Question 1.10 might be rather wild as Γ and i vary.

1.7 Overview

In §2, we relate the definition of graphical groups from §1.1 to that from [29]. Introduced in [29], adjacency modules are modules over polynomial rings whose specialisations are closely related to conjugacy classes of graphical groups. In §3, we determine the dimensions of such specialisations over fields. By combining this with work of O'Brien and Voll [24], in §4, we prove Theorem A. In §5, we show that the polynomials $\mathscr{C}_\Gamma(X, Y)$ are well-behaved with respect to joins of graphs. In §6, we consider the constant term and leading coefficient of $\mathscr{C}_\Gamma(X, Y)$ in Y and we prove Proposition D. Next, §7 is devoted to the degree of $f_\Gamma(X)$. Finally, in §8, we relate our findings to the study of zeta functions enumerating conjugacy classes.

1.8 *Notation*

The symbol "$\subset$" indicates not necessarily proper inclusion. Group commutators are written $[x, y] = x^{-1}y^{-1}xy$. For a ring R and set A, RA denotes the free R-module with basis $(\mathsf{e}_a)_{a\in A}$. For $x \in RA$, we write $x = \sum_{a\in A} x_a \mathsf{e}_a$. We view $d \times e$ matrices over R as maps $R^d \to R^e$ acting by right multiplication. We let $\bullet$ denote a graph with one vertex.

2 Graphical groups and group schemes

Throughout this section, let $\Gamma = (V, E)$ be a graph with n vertices and m edges. We write $V = \{v_1, \dots, v_n\}$ and $J = \{(j, k) : 1 \leqslant j < k \leqslant n, v_j \sim v_k\}$. For a ring R, let $R\,\Gamma$ denote the free R-module of rank $m + n$ with basis consisting of $\mathsf{e}_1, \dots, \mathsf{e}_n$ and all e_{jk} for $(j, k) \in J$. The chosen ordering of V allows us to identify $R\,\Gamma = RV \oplus RE$ (see §1.8).

2.1 Graphical groups over quotients of the integers

Recall that the right-angled Artin group associated with the complement of Γ is

$$\mathsf{A}_\Gamma := \langle x_1, \dots, x_n \mid [x_i, x_j] = 1 \text{ whenever } v_i \not\sim v_j \rangle .$$

Let $\gamma_1(H) \geqslant \gamma_2(H) \geqslant \cdots$ denote the lower central series of a group H. Recall the definition of $\mathbf{G}_\Gamma(R)$ from §1.1.

Proposition 2.1.

(i) (Cf. [29, Rem. 3.8].) $\mathbf{G}_\Gamma(\mathbf{Z}) \approx \mathsf{A}_\Gamma/\gamma_3(\mathsf{A}_\Gamma)$.

(ii) $\mathbf{G}_\Gamma(\mathbf{Z}/N\mathbf{Z}) \approx \mathsf{A}_\Gamma/\gamma_3(\mathsf{A}_\Gamma)\mathsf{A}_\Gamma^N$ *if* $N \geqslant 1$ *is an odd integer.*

Proof. Part (i) follows since $x_i(r) = x_i(1)^r$ and $z_{jk}(r) = z_{jk}(1)^r$ in $\mathbf{G}_\Gamma(\mathbf{Z})$ for $r \in \mathbf{Z}$, $i \in [n]$, and $(j, k) \in J$. Let $G = \langle X \rangle$ be a nilpotent group with $\gamma_3(G) = 1$. As is well known (and easy to see), $[ab, c] = [a, c][b, c]$ and $(ab)^N = a^N b^N [b, a]^{\binom{N}{2}}$ for $a, b, c \in G$; cf. [15, III, Hilfssatz 1.2c) and Hilfssatz 1.3b)]. Let N be odd so that $N \mid \binom{N}{2}$, Then, if $x^N = 1$ for all $x \in X$, we find that $a^N = 1$ for all $a \in G$. Taking $X = \{x_1(1), \dots, x_n(1)\}$ and $G = \mathbf{G}_\Gamma(\mathbf{Z}/N\mathbf{Z})$, we obtain $\mathbf{G}_\Gamma(\mathbf{Z}/N\mathbf{Z}) \approx \mathbf{G}_\Gamma(\mathbf{Z})/\left\langle x_1(1)^N, \dots, x_n(1)^N \right\rangle \approx \mathbf{G}_\Gamma(\mathbf{Z})/\mathbf{G}_\Gamma(\mathbf{Z})^N \approx \mathsf{A}_\Gamma/\gamma_3(\mathsf{A}_\Gamma)\mathsf{A}_\Gamma^N$, which proves (ii). ♦

2.2 Graphical group schemes following [29]

We summarise the construction of the graphical group scheme $\mathbf{H}_\Gamma$ from [29, §3.4] (denoted by $\mathbf{G}_\Gamma$ in [29]). For a ring R, the underlying set of the group $\mathbf{H}_\Gamma(R)$ is $R\Gamma$. The group operation $*$ is characterised as follows:

(G1) $0 \in R\Gamma$ is the identity element of $\mathbf{H}_\Gamma(R)$.

(G2) For all $r_1, \dots, r_n \in R$, we have $r_1\mathsf{e}_1 * \dots * r_n\mathsf{e}_n = r_1\mathsf{e}_1 + \dots + r_n\mathsf{e}_n$.

(G3) For $1 \leqslant i \leqslant j \leqslant n$ and $r, s \in R$, we have

$$s\mathsf{e}_j * r\mathsf{e}_i = \begin{cases} r\mathsf{e}_i + s\mathsf{e}_j - rs\mathsf{e}_{ij}, & \text{if } v_i \sim v_j, \\ r\mathsf{e}_i + s\mathsf{e}_j, & \text{otherwise.} \end{cases}$$

(G4) For all $x \in R\Gamma$ and $z \in RE \subset R\Gamma$, we have $x * z = z * x = x + z$.

Given a ring map $R \to R'$, the induced map $R\Gamma \to R'\Gamma$ is a group homomorphism $\mathbf{H}_\Gamma(R) \to \mathbf{H}_\Gamma(R')$. The resulting group functor $\mathbf{H}_\Gamma$ represents the **graphical group scheme** constructed in [29, §3.4].

2.3 Relating the two constructions of graphical group schemes

The group functors $\mathbf{G}_\Gamma$ (see §1.1) and $\mathbf{H}_\Gamma$ (see §2.2) are naturally isomorphic:

Proposition 2.2. *For each ring R, the map $\theta_R\colon \mathbf{H}_\Gamma(R) \to \mathbf{G}_\Gamma(R)$ given by*

$$\sum_{i=1}^{n} r_i\mathsf{e}_i + \sum_{(j,k)\in J} r_{jk}\mathsf{e}_{jk} \mapsto x_1(r_1)\cdots x_n(r_n) \prod_{(j,k)\in J} z_{jk}(r_{jk}) \qquad (r_i, r_{jk} \in R)$$

is a group isomorphism. These maps combine to form a natural isomorphism of group functors $\mathbf{H}_\Gamma \xrightarrow{\approx} \mathbf{G}_\Gamma$.

Proof. By a simple calculation in $\mathbf{H}_\Gamma(R)$, we find that for $1 \leqslant i < j \leqslant n$ and $r_i, r_j \in R$,

$$[r_i\mathsf{e}_i, r_j\mathsf{e}_j] = \begin{cases} r_ir_j\mathsf{e}_{ij}, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise.} \end{cases}$$

We thus obtain a group homomorphism $\pi_R\colon \mathbf{G}_\Gamma(R) \to \mathbf{H}_\Gamma(R)$ sending each $x_i(r)$ to $r\mathsf{e}_i$ and each $z_{jk}(r)$ to $r\mathsf{e}_{jk}$. By construction, $\pi_R\theta_R = \mathrm{id}_{\mathbf{H}_\Gamma(R)}$ and $\theta_R\pi_R = \mathrm{id}_{\mathbf{H}_\Gamma(R)}$. ♦

We are therefore justified in referring to both $\mathbf{G}_\Gamma$ and $\mathbf{H}_\Gamma$ as “the” graphical group scheme associated with Γ. As a consequence of Proposition 2.2, each $g \in \mathbf{G}_\Gamma(R)$ admits a unique representation

$$g = x_1(r_1)\cdots x_n(r_n) \prod_{(j,k)\in J} z_{jk}(r_{jk}). \qquad (r_i, r_{jk} \in R)$$

In particular, $\mathbf{G}_\Gamma(R)$ has order $|R|^{m+n}$.

2.4 Centralisers in graphical groups and graphical Lie algebras

The **graphical Lie algebra** $\mathfrak{h}_\Gamma(R)$ associated with Γ over a ring R is defined by endowing the module $R\Gamma$ with the Lie bracket $(\cdot,\cdot)$ characterised by the following properties:

- ▷ For $1 \leqslant j < k \leqslant n$, we have $(\mathsf{e}_j, \mathsf{e}_k) = \mathsf{e}_{jk}$ if $(j,k) \in J$ and $(\mathsf{e}_j, \mathsf{e}_k) = 0$ otherwise.
- ▷ For $1 \leqslant i \leqslant n$ and $(j,k),(j',k') \in J$, we have $(\mathsf{e}_i, \mathsf{e}_{jk}) = (\mathsf{e}_{jk}, \mathsf{e}_{j'k'}) = 0$.

We may identify $\mathfrak{h}_\Gamma(R) = \mathfrak{h}_\Gamma(\mathbf{Z}) \otimes R$ as Lie R-algebras and $\mathbf{H}_\Gamma(R) = \mathfrak{h}_\Gamma(R)$ as sets.

Then $\mathbf{H}_\Gamma$ is the group scheme associated with the Lie algebra $\mathfrak{h}_\Gamma(\mathbf{Z})$ via the construction from [32, §2.4.1]; cf. [29, §2.4]. In particular, if $2 \in R^\times$, then $\mathbf{H}_\Gamma(R)$ (and hence $\mathbf{G}_\Gamma(R)$) is isomorphic to the group $\exp(\mathfrak{h}_\Gamma(R))$ associated with $\mathfrak{h}_\Gamma(R)$ via the Lazard correspondence. It follows that for an odd prime p, $\mathbf{H}_\Gamma(\mathbf{F}_p)$ is isomorphic to the finite p-group attached to Γ by Li and Qiao [17]. He and Qiao [13, Thm 1.1] showed that for graphs Γ and Γ' and an odd prime p, $\mathbf{H}_\Gamma(\mathbf{F}_p)$ and $\mathbf{H}_{\Gamma'}(\mathbf{F}_p)$ are isomorphic if and only if Γ and Γ' are.

Proposition 2.3.

(i) *The group centraliser of $h \in R\Gamma$ in $\mathbf{H}_\Gamma(R)$ and the Lie centraliser of h in $\mathfrak{h}_\Gamma(R)$ coincide as sets. Hence, the size of each conjugacy class of $\mathbf{H}_\Gamma(\mathbf{F}_q)$ is a power of q.*

(ii) *The centres of $\mathbf{H}_\Gamma(R)$ and $\mathfrak{h}_\Gamma(R)$ coincide as sets. The centre of $\mathfrak{h}_\Gamma(R)$ is the submodule of $R\Gamma$ generated by all e_{jk} for $(j,k) \in J$ and all e_i for isolated vertices v_i.*

(iii) $[\mathbf{H}_\Gamma(R), \mathbf{H}_\Gamma(R)] = (\mathfrak{h}_\Gamma(R), \mathfrak{h}_\Gamma(R)) = RE$ *and* $\mathbf{H}_\Gamma(R)/[\mathbf{H}_\Gamma(R), \mathbf{H}_\Gamma(R)] \approx RV$.

Proof. The elements $r\mathsf{e}_i$ for $r \in R$ and $i = 1, \dots, n$ generate $\mathbf{H}_\Gamma(R)$ as a group and $\mathfrak{h}_\Gamma(R)$ as a Lie R-algebra. As $\mathbf{H}_\Gamma(R)$ and $\mathfrak{h}_\Gamma(R)$ both have class at most 2, using the calculation from the proof of Proposition 2.2, we find that for $h_1, h_2 \in R\Gamma$, the Lie bracket (h_1, h_2) coincides with the group commutator $[h_1, h_2]$. All claims follow easily from this. ♦

3 Adjacency modules

Let $\Gamma = (V, E)$ be a graph. Let $X_V = (X_v)_{v \in V}$ consist of algebraically independent variables over $\mathbf{Z}$. The **adjacency module** of Γ is the $\mathbf{Z}[X_V]$-module

$$\mathrm{Adj}(\Gamma) := \frac{\mathbf{Z}[X_V]V}{\langle X_v\mathsf{e}_w - X_w\mathsf{e}_v : v, w \in V \text{ with } v \sim w\rangle}.$$

These modules were introduced in [29, §3.3]. Their study turns out to be closely related to the enumeration of conjugacy classes of graphical groups; see [29, §§3.4,6,7]. We note that what we call adjacency modules here are dubbed *negative* adjacency modules in [29].

For a ring R and $x \in RV$, we obtain an R-module by specialising $\mathrm{Adj}(\Gamma)$ in the form

$$\mathrm{Adj}(\Gamma)_x := \mathrm{Adj}(\Gamma) \otimes_{\mathbf{Z}[X_V]} R_x \approx \frac{RV}{\langle x_v\mathsf{e}_w - x_w\mathsf{e}_v : v, w \in V \text{ with } v \sim w\rangle},$$

where R_x denotes R regarded as a $\mathbf{Z}[X_V]$-algebra via $X_v r = x_v r$ for $v \in V$ and $r \in R$.

Lemma 3.1. *Let $\Gamma_i = (V_i, E_i)$ $(i = 1,2)$ be graphs on disjoint vertex sets. Let $\Gamma = \Gamma_1 \oplus \Gamma_2 = (V,E)$ be their disjoint union. Let R be a ring and let $x \in RV$. Let $x_i \in RV_i$ denote the image of x under the natural projection $RV = RV_1 \oplus RV_2 \to RV_i$. Then $\mathrm{Adj}(\Gamma)_x \approx \mathrm{Adj}(\Gamma_1)_{x_1} \oplus \mathrm{Adj}(\Gamma_2)_{x_2}$ as R-modules.* ♦

Let K be a field. For $x \in KV$, let $\mathrm{supp}(x) = \{v \in V : x_v \neq 0\}$. Recall the definitions of $\mathrm{N}_\Gamma[U]$ and $\mathrm{c}_\Gamma(U)$ from §1. The following is a key ingredient of our proof of Theorem A.

Lemma 3.2. *Let $x \in KV$ and $U = \mathrm{supp}(x)$. Then*

$$\dim(\mathrm{Adj}(\Gamma)_x) = \mathrm{c}_\Gamma(U) + |V| - |\mathrm{N}_\Gamma[U]|.$$

Proof. Let $H := \langle x_v \mathsf{e}_w - x_w \mathsf{e}_v : v \sim w \rangle \leqslant KV$ so that $\mathrm{Adj}(\Gamma)_x \approx KV/H$.

(a) Suppose that Γ is connected and $U = V$. We need to show that $\dim(\mathrm{Adj}(\Gamma)_x) = 1$. To see that, first note that $H \subset x^\perp$, where the orthogonal complement is taken with respect to the bilinear form $y \cdot z = \sum_{v \in V} y_v z_v$. Hence, $\mathrm{Adj}(\Gamma)_x \neq 0$. Choose a spanning tree T of Γ and a root $r \in V$. For $v \in V \setminus \{r\}$, let $\mathrm{p}(v)$ be the predecessor of v on the unique path from r to v in T. As the elements $\mathsf{e}_v - \frac{x_v}{x_{\mathrm{p}(v)}} \mathsf{e}_{\mathrm{p}(v)} \in H$ for $v \in V \setminus \{r\}$ are linearly independent, $\dim(\mathrm{Adj}(\Gamma)_x) \leqslant 1$. Thus, $\dim(\mathrm{Adj}(\Gamma)_x) = 1$.

(b) If $U = V$ but Γ is possibly disconnected, then (a) and Lemma 3.1 show that $\dim(\mathrm{Adj}(\Gamma)_x) = \mathrm{c}_\Gamma(U)$ is the number of connected components of Γ, as claimed.

(c) For the general case, let $x[U] := \sum_{u \in U} x_u \mathsf{e}_u \in KU$ be the image of x under the natural projection $KV = KU \oplus K(V \setminus U) \to KU$. We claim that $\mathrm{Adj}(\Gamma)_x \approx \mathrm{Adj}(\Gamma[U])_{x[U]} \oplus K(V \setminus \mathrm{N}_\Gamma[U])$. Indeed, this follows since H is spanned by the following two types of elements:

 ▷ $x_u \mathsf{e}_v - x_v \mathsf{e}_u$ for adjacent vertices $u, v \in U$.

 ▷ e_w for $w \in \mathrm{N}_\Gamma[U] \setminus U$.

The claim follows since by (b), $\dim(\mathrm{Adj}(\Gamma[U])_{x[U]}) = \mathrm{c}_\Gamma(U)$. ♦

Remark 3.3. Let Γ have n vertices and c connected components. Lemma 3.2 generalises a well-known basic fact: each oriented incidence matrix of Γ has rank $\mathrm{rk}(\Gamma) = n - c$; see [4, Prop. 4.3]. This easily implies the special case $x = \sum_{v \in V} \mathsf{e}_v$ of Lemma 3.2.

4 Proof of Theorem A

We first rephrase Theorem A. For a finite group G, define a Dirichlet polynomial $\zeta_G^{\mathrm{cc}}(s) = \sum_{e=1}^{\infty} \mathrm{cc}_e(G) e^{-s}$; here, s denotes a complex variable. For almost simple groups, these functions were studied in [18]. Following [19], we refer to $\zeta_G^{\mathrm{cc}}(s)$ as the **conjugacy class zeta function** of G. We note that a different notion of conjugacy class zeta functions, occasionally denoted using the same notation $\zeta_G^{\mathrm{cc}}(s)$, can also be found in the literature; see [3, 7, 27, 28]. Following [29], in §8.1, we will refer to the latter functions as *class-counting zeta functions*.

Let $\Gamma = (V,E)$ be a graph with n vertices and m edges. Theorem A is equivalent to $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma(\mathbf{F}_q)}(s) = q^m \mathscr{C}_\Gamma(q, q^{-1-s})$.

Lemma 4.1. $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma(\mathbf{F}_q)}(s) = q^{m-n(s+1)} \sum_{x\in\mathbf{F}_q V} |\mathrm{Adj}(\Gamma)_x|^{s+1}$.

Proof. Write $V = \{v_1,\dots,v_n\}$ and $J = \{(j,k) : 1 \leqslant j < k \leqslant n \text{ with } v_j \sim v_k\}$; for a ring R, we identify $RV = R^n$. We assume that $v_{n'+1},\dots,v_n$ are the isolated vertices of Γ. Order the elements of J lexicographically to establish a bijection between $\{1,\dots,m\}$ and J.

Write $\mathfrak{h} = \mathfrak{h}_\Gamma(\mathbf{Z})$; see §2.4. Let $\mathfrak{h}'$ and $\mathfrak{z}$ denote the derived subalgebra and centre of $\mathfrak{h}$, respectively. By Proposition 2.3, $\mathfrak{h}'$ and $\mathfrak{z}$ are free $\mathbf{Z}$-modules of ranks m and $m+n-n'$, respectively. Moreover, the images of $\mathsf{e}_1,\dots,\mathsf{e}_{n'}$ form a $\mathbf{Z}$-basis of $\mathfrak{h}/\mathfrak{z}$. Proposition 2.3 also shows that for each ring R, we may identify $\mathfrak{h}'\otimes R$ with the derived subalgebra of $\mathfrak{h}\otimes R$, and $\mathfrak{z}\otimes R$ with the centre of $\mathfrak{h}\otimes R$.

Suppose that $q = p^f$ for an *odd* prime p. As we noted in §2.4, $\mathbf{G}_\Gamma(\mathbf{F}_q)$ is isomorphic to the group $\exp(\mathfrak{h}_\Gamma(\mathbf{F}_q))$ attached to the Lie $\mathbf{F}_q$-algebra $\mathfrak{h}_\Gamma(\mathbf{F}_q) = \mathfrak{h}\otimes\mathbf{F}_q$ via the Lazard correspondence. Let $A(X_1,\dots,X_{n'}) \in \mathrm{M}_{n'\times m}(\mathbf{Z}[X_1,\dots,X_{n'}])$ be the matrix of linear forms whose (j,k)th column has precisely two non-zero entries, namely X_k and $-X_j$ in rows j and k, respectively. Let $\mathbf{Z}_p$ denote the ring of p-adic integers. It is readily verified that the image of the matrix $A(X_1,\dots,X_{n'})$ over $\mathbf{Z}_p[X_1,\dots,X_{n'}]$ is a “commutator matrix” (as defined in [24, Def. 2.1]) associated with the finite Lie $\mathbf{Z}_p$-algebra $\mathfrak{h}\otimes\mathbf{F}_p$.

By [24, Thm B]

$$\mathrm{cc}_{q^i}(\mathbf{G}_\Gamma(\mathbf{F}_q)) = \#\{x \in \mathbf{F}_q^{n'} : \mathrm{rk}_{\mathbf{F}_q}(A(x)) = i\}\cdot q^{n-n'+m-i}.$$

Let $\breve{A}(X)$ be the $m\times n$ matrix over $\mathbf{Z}[X] = \mathbf{Z}[X_1,\dots,X_n]$ which is obtained from $A(X_1,\dots,X_{n'})^\top$ by adding zero columns in positions $n'+1,\dots,n$. Hence,

$$\mathrm{cc}_{q^i}(\mathbf{G}_\Gamma(\mathbf{F}_q)) = \#\{x \in \mathbf{F}_q^{n} : \mathrm{rk}_{\mathbf{F}_q}(\breve{A}(x)) = i\}\cdot q^{m-i}.$$

By construction, $\mathrm{Adj}(\Gamma)_x \approx \mathrm{Coker}(\breve{A}(x))$ for all $x \in \mathbf{F}_qV = \mathbf{F}_q^n$. In particular, for $x\in\mathbf{F}_q^n$, we have $\mathrm{rk}_{\mathbf{F}_q}(\breve{A}(x)) = i$ if and only if $\dim_{\mathbf{F}_q}(\mathrm{Adj}(\Gamma)_x) = n-i$. Hence, writing $\alpha_x = |\mathrm{Adj}(\Gamma)_x|$, we have $\mathrm{rk}_{\mathbf{F}_q}(\breve{A}(x)) = i$ if and only if $q^n\alpha_x^{-1} = q^i$. Thus,

$$\begin{aligned}\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma(\mathbf{F}_q)}(s) &= \sum_{i=0}^{\infty}\mathrm{cc}_{q^i}(\mathbf{G}_\Gamma(\mathbf{F}_q))q^{-is} = \sum_{x\in\mathbf{F}_qV} q^{m-n}\alpha_x\cdot(q^n\alpha_x^{-1})^{-s}\\ &= q^{m-n(s+1)}\sum_{x\in\mathbf{F}_qV}\alpha_x^{s+1}.\end{aligned}$$

Finally, if q is even, while the statement of [24, Thm B] itself is no longer directly applicable (due to its reliance on the Lazard correspondence), its proof in [24, §§3.1, 3.3–3.4] *does* apply in the present setting, completing the present proof. Indeed, the key ingredient that we need is to be able to identify $\mathbf{G}_\Gamma(\mathbf{F}_q)$ and $\mathfrak{h}\otimes\mathbf{F}_q$ as sets such that two elements commute in the group if and only if they commute in the Lie algebra. These conditions are satisfied by Propositions 2.2–2.3. ⧫

Proof of Theorem A. By combining Lemma 4.1 and Lemma 3.2, we obtain

$$\begin{aligned}\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma(\mathbf{F}_q)}(s) &= q^{m-n(s+1)} \sum_{x\in\mathbf{F}_q V} \Big(q^{\mathrm{c}_\Gamma(\mathrm{supp}(x))+n-|\mathrm{N}_\Gamma[\mathrm{supp}(x)]|}\Big)^{s+1}\\ &= q^m \sum_{x\in\mathbf{F}_q V} (q^{-1-s})^{|\mathrm{N}_\Gamma[\mathrm{supp}(x)]|-\mathrm{c}_\Gamma(\mathrm{supp}(x))}\\ &= q^m \sum_{U\subset V} (q-1)^{|U|}(q^{-1-s})^{|\mathrm{N}_\Gamma[U]|-\mathrm{c}_\Gamma(U)}\\ &= q^m \mathscr{C}_\Gamma(q, q^{-1-s}).\end{aligned}$$ ♦

5 Graph operations: disjoint unions and joins

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be graphs with $V_1 \cap V_2 = \emptyset$. Let Γ_i have n_i vertices and m_i edges. The disjoint union $\Gamma_1 \oplus \Gamma_2$ and join $\Gamma_1 \vee \Gamma_2$ (see §1.2) of Γ_1 and Γ_2 are both graphs on the vertex set $V_1 \cup V_2$ with $m_1 + m_2$ and $m_1 + m_2 + n_1 n_2$ edges, respectively.

Proposition 5.1. $\mathscr{C}_{\Gamma_1\oplus\Gamma_2}(X, Y) = \mathscr{C}_{\Gamma_1}(X, Y)\mathscr{C}_{\Gamma_2}(X, Y)$.

Proof. This follows since if $U_i \subset V_i$ for $i = 1, 2$, then $\mathrm{N}_{\Gamma_1\oplus\Gamma_2}[U_1 \cup U_2] = \mathrm{N}_{\Gamma_1}[U_1] \cup \mathrm{N}_{\Gamma_2}[U_2]$ and $\mathrm{c}_{\Gamma_1\oplus\Gamma_2}(U_1 \cup U_2) = \mathrm{c}_{\Gamma_1}(U_1) + \mathrm{c}_{\Gamma_2}(U_2)$. ♦

Proposition 5.1 also follows, a fortiori, from Theorem A and the identity $\mathrm{cc}_e(G_1 \times G_2) = \sum_{d|e} \mathrm{cc}_d(G_1)\,\mathrm{cc}_{e/d}(G_2)$ for finite groups G_1 and G_2.

Proposition 5.2.

$$\mathscr{C}_{\Gamma_1\vee\Gamma_2}(X, Y) = 1 + \Big(\mathscr{C}_{\Gamma_1}(X, Y) - 1\Big)Y^{n_2} + Y^{n_1}\Big(\mathscr{C}_{\Gamma_2}(X, Y) - 1\Big) + (X^{n_1} - 1)(X^{n_2} - 1)Y^{n_1+n_2-1}.$$

Proof. Write $\Gamma = \Gamma_1 \vee \Gamma_2$ and $V = V_1 \cup V_2$. Let $U_i \subset V_i$ for $i = 1, 2$ and $U = U_1 \cup U_2$. We seek to relate the summand $t(U) := (X - 1)^{|U|}Y^{|\mathrm{N}_\Gamma[U]|-\mathrm{c}_\Gamma(U)}$ in the definition of $\mathscr{C}_\Gamma(X, Y)$ to the summands $t_i(U_i) := (X - 1)^{|U_i|}Y^{|\mathrm{N}_{\Gamma_i}[U_i]|-\mathrm{c}_{\Gamma_i}(U_i)}$. We consider four cases:

1. If $U_1 = U_2 = U = \emptyset$, then $t(U) = 1$.
2. If $U_1 \neq \emptyset = U_2$, then $\mathrm{N}_\Gamma[U] = \mathrm{N}_{\Gamma_1}[U_1] \cup V_2$, $\mathrm{c}_\Gamma(U) = \mathrm{c}_{\Gamma_1}(U_1)$, and $t(U) = t_1(U_1)Y^{n_2}$.
3. Analogously, if $U_1 = \emptyset \neq U_2$, then $t(U) = Y^{n_1}t_2(U_2)$.
4. If $U_1 \neq \emptyset \neq U_2$, then $\mathrm{N}_\Gamma[U] = V$, $\mathrm{c}_\Gamma(U) = 1$, and $t(U) = (X - 1)^{|U_1|+|U_2|}Y^{n_1+n_2-1}$.

We conclude that

$$\begin{aligned}\mathscr{C}_\Gamma(X, Y) = 1 &+ \Big(\mathscr{C}_{\Gamma_1}(X, Y) - 1\Big)Y^{n_2} + Y^{n_1}\Big(\mathscr{C}_{\Gamma_2}(X, Y) - 1\Big)\\ &+ \left(\sum_{\substack{\emptyset\neq U_1\subset V_1\\ \emptyset\neq U_2\subset V_2}} (X - 1)^{|U_1|}(X - 1)^{|U_2|}\right)Y^{n_1+n_2-1}.\end{aligned}$$ ♦

As we will explain in §8.2, Proposition 5.2 is closely related to [29, Prop. 8.4].

Example 5.3 (Complete bipartite graphs)**.** Let $\mathrm{K}_{a,b} = \Delta_a \vee \Delta_b$ be a complete bipartite graph. Recall from Example 1.7 that $\mathscr{C}_{\Delta_n}(X,Y) = X^n$. Therefore, by Proposition 5.2, $\mathscr{C}_{\mathrm{K}_{a,b}}(X,Y) = 1 + (X^a-1)Y^b + Y^a(X^b-1) + (X^a-1)(X^b-1)Y^{a+b-1}$. Hence,

$$\begin{aligned}\mathsf{F}_{\mathrm{K}_{a,b}}(X,Y) = X^{(a-1)(b-1)}&(X^a-1)(X^b-1)Y^{a+b-1} \\ &+ X^{(a-1)b}(X^a-1)Y^b + X^{a(b-1)}(X^b-1)Y^a + X^{ab}\end{aligned}$$

and $f_{\mathrm{K}_{a,b}}(X) = X^{(a-1)(b-1)}((X^a-1)(X^b-1) + X^{a-1}(X^a-1) + X^{b-1}(X^b-1) + X^{a+b-1})$. We note that the graphical group $\mathbf{G}_{\mathrm{K}_{a,b}}(\mathbf{Z}/N\mathbf{Z})$ is the maximal quotient of class at most 2 of the free product $(\mathbf{Z}/N\mathbf{Z})^a * (\mathbf{Z}/N\mathbf{Z})^b$; see [29, §3.4].

Example 5.4 (Stars)**.** As a special case of Example 5.3, let $\mathrm{Star}_n = \Delta_n \vee \bullet = \mathrm{K}_{n,1}$ be a star graph on $n+1$ vertices. Then $\mathscr{C}_{\mathrm{Star}_n}(X,Y) = (X^{n+1}-X^n)Y^n + (X^n-1)Y + 1$. Hence, $\mathsf{F}_{\mathrm{Star}_n}(X,Y) = X^{n-1}\cdot((X^2-X)Y^n + (X^n-1)Y + X)$ and $f_{\mathrm{Star}_n}(X) = X^{n-1}(X^n + X^2 - 1)$.

We record the following consequence of Proposition 5.2 for later use.

Corollary 5.5. *Let Γ_1 and Γ_2 be graphs. Let Γ_i have m_i edges and n_i vertices. Then*

$$\begin{aligned}f_{\Gamma_1\vee\Gamma_2}(X) = X^{m_1+m_2+n_1n_2} &+ X^{m_2+(n_1-1)n_2}(f_{\Gamma_1}(X) - X^{m_1}) \\ &+ X^{m_1+n_1(n_2-1)}(f_{\Gamma_2}(X) - X^{m_2}) \\ &\quad + X^{m_1+m_2+(n_1-1)(n_2-1)}(X^{n_1}-1)(X^{n_2}-1). \qquad (5.1)\end{aligned}$$

♦

Beyond disjoint unions and joins, it would be natural to study the effects of other graph operations on the polynomials $\mathscr{C}_\Gamma(X,Y)$.

6 The constant and leading term of $\mathscr{C}_\Gamma(X,Y)$

Let $\Gamma = (V,E)$ be a graph with n vertices and m edges. In this section, we primarily view $\mathscr{C}_\Gamma(X,Y)$ as a polynomial in Y over $\mathbf{Z}[X]$. Its constant term is easily determined.

Proposition 6.1. *$\mathscr{C}_\Gamma(X,0) = X^i$, where i is the number of isolated vertices of Γ.*

Proof. As $\mathscr{C}_\bullet(X,Y) = X$, by Proposition 5.1, $\mathscr{C}_{\Gamma\oplus\bullet}(X,Y) = X\cdot\mathscr{C}_\Gamma(X,Y)$. We may thus assume that $i=0$. Let $U \subset V$. As $\mathrm{c}_\Gamma(U) \leqslant |U| \leqslant |\mathrm{N}_\Gamma[U]|$, we see that $|\mathrm{N}_\Gamma[U]| = \mathrm{c}_\Gamma(U)$ if and only if U consists of isolated vertices. This only happens for $U = \emptyset$ whence $\mathscr{C}_\Gamma(X,0) = 1$. ♦

For a group-theoretic interpretation of Proposition 6.1, note that q^i is the order of the quotient $\mathrm{Z}(\mathbf{G}_\Gamma(\mathbf{F}_q))/[\mathbf{G}_\Gamma(\mathbf{F}_q),\mathbf{G}_\Gamma(\mathbf{F}_q)]$.

Recall that $\mathrm{rk}(\Gamma) = n - c$, where c is the number of connected components of Γ.

Proposition 6.2. $\deg_Y(\mathscr{C}_\Gamma(X,Y)) = \mathrm{rk}(\Gamma)$.

Proof. If Γ' is any subgraph of Γ, then $\mathrm{rk}(\Gamma') \leqslant \mathrm{rk}(\Gamma)$. Let $U \subset V$ and write $\bar{U} = \mathrm{N}_\Gamma[U]$. Since every vertex in $\bar{U}\setminus U$ is adjacent to some vertex in U, we have $\mathrm{c}_\Gamma(\bar{U}) \leqslant \mathrm{c}_\Gamma(U)$. Hence, $|\bar{U}| - \mathrm{c}_\Gamma(U) \leqslant |\bar{U}| - \mathrm{c}_\Gamma(\bar{U}) = \mathrm{rk}(\Gamma[\bar{U}]) \leqslant \mathrm{rk}(\Gamma)$. Thus, $\deg_Y(\mathscr{C}_\Gamma(X,Y)) \leqslant \mathrm{rk}(\Gamma)$. The summand corresponding to $U = V$ in (1.2) contributes a term $X^nY^{\mathrm{rk}(\Gamma)}$ to $\mathscr{C}_\Gamma(X,Y)$, and this term cannot be cancelled by a summand arising from any proper subset. ♦

For $h(X,Y) = \sum_{ij} a_{ij} X^i Y^j \in \mathbf{Z}[X,Y]$ with $a_{ij} \in \mathbf{Z}$, write $h(X,Y)\left[Y^j\right] = \sum_i a_{ij} X^i$ for the coefficient of Y^j in $h(X,Y)$, regarded as a polynomial in Y. We now consider the leading coefficient $\mathscr{C}_\Gamma(X,Y)\left[Y^{\mathrm{rk}(\Gamma)}\right]$ of $\mathscr{C}_\Gamma(X,Y)$ as a polynomial in Y. Recall that a **dominating set** of Γ is a set $D \subset V$ with $\mathrm{N}_\Gamma[D] = V$. If, in addition, $\Gamma[D]$ is connected, then D is a **connected dominating set**. Let $\mathfrak{D}^{\mathrm{c}}(\Gamma)$ be the set of connected dominating sets of Γ. Clearly, $\mathfrak{D}^{\mathrm{c}}(\Gamma) \neq \emptyset$ if and only if Γ is connected.

Proposition 6.3. *Suppose that $n \geqslant 2$. Then*

$$\mathscr{C}_\Gamma(X+1,Y)\left[Y^{n-1}\right] = \sum_{D \in \mathfrak{D}^{\mathrm{c}}(\Gamma)} X^{|D|}. \tag{6.1}$$

Proof. Let $U \subset V$. As $n \geqslant 2$, $|\mathrm{N}_\Gamma[U]| - \mathrm{c}_\Gamma(U) = n-1$ if and only if $\mathrm{N}_\Gamma[U] = V$ and $\mathrm{c}_\Gamma(U) = 1$. The latter two conditions are satisfied if and only if $U \in \mathfrak{D}^{\mathrm{c}}(\Gamma)$. ♦

Remark 6.4. In [23], the right-hand side of (6.1) is referred to as the **connected domination polynomial** of Γ. These polynomials are relatives of the widely studied domination polynomials of graphs introduced in [1] (where they were called dominating polynomials).

Corollary 6.5. *Suppose that Γ does not contain isolated vertices. Let $V_1,\dots,V_c \subset V$ be the distinct connected components of Γ. Then*

$$\mathscr{C}_\Gamma(X+1,Y)\left[Y^{\mathrm{rk}(\Gamma)}\right] = \prod_{i=1}^{c} \sum_{D_i \in \mathfrak{D}^{\mathrm{c}}(\Gamma[V_i])} X^{|D_i|}. \qquad ♦$$

The following is well known.

Theorem 6.6 ([9, §A1.1, [GT2]]). *The problem of deciding, for a given graph Γ and $k \geqslant 1$, whether Γ admits a connected dominating set of cardinality at most k is NP-complete.*

Proof of Proposition D. Combine Theorem 6.6 and Proposition 6.3. ♦

We finish this section by showing that typically $\mathsf{F}_\Gamma(0,Y) = 0$. We first record the following consequence of Proposition 6.3.

Corollary 6.7. *Let Γ be a tree with $n \geqslant 3$ vertices and ℓ leaves. Then $\mathscr{C}_\Gamma(X,Y)\left[Y^{n-1}\right] = (X-1)^{n-\ell} X^\ell$.*

Proof. Let $V^{\text{leaf}} \subset V$ be the set of leaves of Γ. Using $n \geqslant 3$, it is easy to see that $\mathfrak{D}^{\mathrm{c}}(\Gamma) = \{U : V \setminus V^{\text{leaf}} \subset U \subset V\}$. Hence, by by Proposition 6.3, $\mathscr{C}_\Gamma(X,Y)\left[Y^{n-1}\right] = (X-1)^{n-\ell} \sum_{U' \subset V^{\text{leaf}}} (X-1)^{|U'|} = (X-1)^{n-\ell} X^\ell$. ♦

Corollary 6.8. *Let Γ be an arbitrary graph. Then $\mathsf{F}_\Gamma(0,Y) = 0$ unless $\Gamma \approx \mathrm{K}_2^{\oplus r}$, in which case $\mathsf{F}_\Gamma(0,Y) = (-1)^r Y^r$.*

Proof. Using Proposition 5.1 (and its evident analogue for $\mathsf{F}_\Gamma(X,Y)$), we may assume that Γ is connected. If $m > \mathrm{rk}(\Gamma)$, then X divides $\mathsf{F}_\Gamma(X,Y)$ by Proposition 6.2. Thus, suppose that $m = \mathrm{rk}(\Gamma) = n-1$, i.e. Γ is a tree. Since $\mathsf{F}_{\mathrm{K}_1}(X,Y) = X$ and $\mathsf{F}_{\mathrm{K}_2}(X,Y) = (X^2-1)Y + X$, we may assume that $n \geqslant 3$. Corollary 6.7 then implies that $\mathsf{F}_\Gamma(0,Y) = 0$. ♦

Remark 6.9. The constant term and leading coefficient of $\mathscr{C}_\Gamma(X,Y)$ as a polynomial in $X-1$ are easily determined: $\mathscr{C}_\Gamma(1,Y)=1$ and $\mathscr{C}_\Gamma(X+1,Y)=X^nY^{\mathrm{rk}(\Gamma)}+\mathcal{O}(X^{n-1})$. The constant term of $\mathscr{C}_\Gamma(X,Y)$ in X, i.e. the polynomial $\mathscr{C}_\Gamma(0,Y)=\sum_{U\subset V}(-1)^{|U|}Y^{|\mathrm{N}_\Gamma[U]|-\mathrm{c}_\Gamma(U)}$, seems to be more mysterious.

7 The degrees of class-counting polynomials

In this section, we consider the degrees of class-counting polynomials $f_\Gamma(X)=\mathsf{F}_\Gamma(X,1)$ (see Theorem 1.2 and Corollary B). As before, let $\Gamma=(V,E)$ be a graph with m edges and n vertices.

7.1 Interpreting $\deg(f_\Gamma(X))$: the invariant $\eta(\Gamma)$

For $U\subset V$, let $\mathrm{d}_\Gamma(U)=|\mathrm{N}_\Gamma[U]\setminus U|$, the number of vertices in $V\setminus U$ with a neighbour in U. Recall that $\mathrm{c}_\Gamma(U)$ denotes the number of connected components of $\Gamma[U]$. Define

$$\eta(\Gamma)=\max_{U\subset V}\Big(\mathrm{c}_\Gamma(U)-\mathrm{d}_\Gamma(U)\Big)\geqslant 0. \tag{7.1}$$

Corollary B implies

$$\deg(f_\Gamma(X))=m+\eta(\Gamma). \tag{7.2}$$

Our proof of Proposition D does not imply that computing $f_\Gamma(X)=X^m\mathscr{C}_\Gamma(X,X^{-1})$ is NP-hard, motivating Question 1.9.

Question 7.1. Is there a polynomial-time algorithm for computing $\eta(\Gamma)$?

Remark 7.2. The author is unaware of previous investigations of the numbers $\eta(\Gamma)$ in the literature. At a formal level, $\eta(\Gamma)$ is reminiscent of other graph-theoretic invariants such as *critical independence numbers* [37] (which can be computed in polynomial time).

In the following, we establish bounds for $\eta(\Gamma)$. Let $\alpha(\Gamma)$ denote the **independence number** of Γ, i.e. the maximal cardinality of an independent set of vertices. Clearly,

$$\eta(\Gamma)\leqslant\max_{U\subset V}\mathrm{c}_\Gamma(U)=\alpha(\Gamma). \tag{7.3}$$

While $\eta(\Gamma)$ can be much smaller than $\alpha(\Gamma)$ (cf. Proposition 7.5(ii)), the bound $\eta(\Gamma)\leqslant\alpha(\Gamma)$ will be useful in our proof of Proposition 7.8 below.

Let c be the number of connected components of Γ. The case $U=V$ in (7.1) shows that $\eta(\Gamma)\geqslant c$. Since $\eta(\Gamma_1\oplus\Gamma_2)=\eta(\Gamma_1)+\eta(\Gamma_2)$, we may assume that Γ is connected.

Proposition 7.3. *Let Γ be connected and $n\geqslant 4$. Then $\eta(\Gamma)\leqslant n-2$ with equality if and only if $\Gamma\approx\mathrm{Star}_{n-1}$.*

Proof. For $U\in\{\emptyset,V\}$, we have $\mathrm{c}_\Gamma(U)-\mathrm{d}_\Gamma(U)\leqslant 1<n-2$. Let $U\subset V$ with $\emptyset\neq U\neq V$. Then $\mathrm{c}_\Gamma(U)\leqslant n-1$ and $\mathrm{d}_\Gamma(U)>0$ since Γ is connected. Hence, $\mathrm{c}_\Gamma(U)-\mathrm{d}_\Gamma(U)\leqslant n-2$ and $\eta(\Gamma)\leqslant n-2$. Moreover, if $\mathrm{c}_\Gamma(U)-\mathrm{d}_\Gamma(U)=n-2$, then $\mathrm{c}_\Gamma(U)=n-1$ and $\mathrm{d}_\Gamma(U)=1$. This is equivalent to Γ being a star graph whose centre is the unique vertex in $V\setminus U$. ⧫

By Proposition 7.3, $\eta(\Gamma)$ rarely attains its maximal value among graphs with n vertices. In contrast, $\eta(\Gamma) = 1$ occurs frequently. Note that $\eta(\mathrm{K}_n) = 1$ by Example 1.3. For complete bipartite graphs, we obtain the following.

Proposition 7.4. $\eta(\mathrm{K}_{a,b}) = \max(1, |a-b|)$.

Proof. This follows by inspection from the formula for $f_{\mathrm{K}_{a,b}}(X)$ in Example 5.3. ⧫

Hence, $\eta(\mathrm{K}_{a,b}) = 1$ if and only if $|a-b| \leqslant 1$. To obtain further examples of graphs Γ with $\eta(\Gamma) = 1$, recall that a graph is **claw-free** if it does not contain $\mathrm{K}_{1,3} \approx \mathrm{Star}_3$ as an induced subgraph. The following proposition and its proof are due to Matteo Cavaleri. The author thanks him for kindly permitting this material to be included here.

Proposition 7.5.

(i) $\eta(\Gamma) = \max\Big(\mathrm{c}_\Gamma(U) + |U| - |V| : U \subset V$ *is a dominating set of* $\Gamma\Big)$.

(ii) If Γ *is claw-free and connected, then* $\eta(\Gamma) = 1$ *and thus* $\deg(f_\Gamma(X)) = m + 1$.

Proof.

(i) Let $U \subset V$ with $\mathrm{N}_\Gamma[U] \neq V$. Let $C \subset V$ be a connected component of $\Gamma[V \setminus \mathrm{N}_\Gamma[U]]$. Clearly, $\mathrm{c}_\Gamma(U \cup C) = \mathrm{c}_\Gamma(U) + 1$. Let $x \in \mathrm{N}_\Gamma[U \cup C] \setminus (U \cup C)$. Then $x \sim y$ for some $y \in U \cup C$. Suppose that $x \notin \mathrm{N}_\Gamma[U]$ so that $y \in C$. Then $x \in C$ by the definition of C. This contradiction shows that $\mathrm{N}_\Gamma[U \cup C] \setminus (U \cup C) \subset \mathrm{N}_\Gamma[U] \setminus U$. Hence, $\mathrm{c}_\Gamma(U \cup C) - \mathrm{d}_\Gamma(U \cup C) > \mathrm{c}_\Gamma(U) - \mathrm{d}_\Gamma(U)$. It follows that the maximal value of $\mathrm{c}_\Gamma(U) - \mathrm{d}_\Gamma(U)$ is attained for a dominating set U; in that case, $\mathrm{d}_\Gamma(U) = |V| - |U|$.

(ii) Let $U \subset V$ be a $\subset$-maximal dominating set with $\mathrm{c}_\Gamma(U) + |U| - |V| = \eta(\Gamma)$. Suppose that $U \neq V$. Choose $x \in V \setminus U$. By maximality of U, $\mathrm{c}_\Gamma(U) + |U| > \mathrm{c}_\Gamma(U \cup \{x\}) + |U| + 1$. Hence, there are distinct connected components C_1, C_2, C_3 of $\Gamma[U]$ such that $\Gamma[C_1 \cup C_2 \cup C_3 \cup \{x\}]$ is connected. Choose $c_i \in C_i$ with $x \sim c_i$. Then $\Gamma[\{c_1, c_2, c_3, x\}] \approx \mathrm{Star}_3$. We conclude that if Γ is claw-free and connected, then $U = V$ and thus $\eta(\Gamma) = 1$. ⧫

Let $\Delta(\Gamma)$ denote the maximum vertex degree of Γ.

Lemma 7.6. *Let* T *be a tree. Then* $\eta(\mathsf{T}) \geqslant \Delta(\mathsf{T}) - 1$.

Proof. Let $w_1, \dots, w_d$ be the distinct vertices adjacent to a vertex u of T. Let W_i consist of w_i and all its descendants in the rooted tree (T, u). Define $W := W_1 \cup \dots \cup W_d$. By construction, $\mathrm{c}_\mathsf{T}(W) = d$ and $\mathrm{d}_\mathsf{T}(W) = 1$ whence $\eta(\mathsf{T}) \geqslant d - 1$. ⧫

Corollary 7.7. *Let* T *be a tree. Then* $\eta(\mathsf{T}) = 1$ *if and only if* T *is a path.*

Proof. By Lemma 7.6, $\eta(\mathsf{T}) > 1$ unless T is a path. By Proposition 7.5(ii) or equation (1.1), we have $\eta(\mathrm{P}_n) = 1$. ⧫

7.2 Upper and lower bounds for $\deg(f_\Gamma(X))$

We obtain sharp bounds for $\deg(f_\Gamma(X))$ as Γ ranges over all graphs with n vertices.

Proposition 7.8. *Let Γ be a graph with $n \geqslant 1$ vertices. Then $n \leqslant \deg(f_\Gamma(X)) \leqslant \binom{n}{2} + 1$. The lower bound is attained if and only if Γ is a disjoint union of paths. The upper bound is attained if and only if Γ is complete or $n = 2$.*

Our proof of Proposition 7.8 will rely on an upper bound for independence numbers.

Lemma 7.9 ([12])**.** *Let Γ be a graph with m edges and n vertices. Then*

$$\alpha(\Gamma) \leqslant \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2m} \right\rfloor .$$

Proof of Proposition 7.8. As before, let m denote the number of edges of $\Gamma = (V, E)$.

(i) *Lower bound.* We may assume that Γ is connected so that $m \geqslant n - 1$. As $\eta(\Gamma) \geqslant 1$, equation (7.2) shows that $\deg(f_\Gamma(X)) \geqslant n$ with equality if and only if Γ is a tree and $\eta(\Gamma) = 1$. By Corollary 7.7, the latter condition is equivalent to $\Gamma \approx \mathrm{P}_n$.

(ii) *Upper bound.* Since $\deg(f_{\mathrm{K}_n}(X)) = \binom{n}{2} + 1$ by Example 1.3, it suffices to show that $\deg(f_\Gamma(X)) \leqslant \binom{n}{2}$ whenever $m < \binom{n}{2}$. We may assume that $n \geqslant 3$. Writing $m = \binom{n}{2} - k$, Lemma 7.9 shows that $\alpha(\Gamma) \leqslant \left\lfloor \frac{1}{2} + \sqrt{2k + \frac{1}{4}} \right\rfloor$. Hence, if $k \geqslant 2$, then $\alpha(\Gamma) \leqslant k$. By equation (7.3), $\deg(f_\Gamma(X)) = m + \eta(\Gamma) \leqslant m + \alpha(\Gamma) \leqslant m + k = \binom{n}{2}$. For $k = 1$, we have $\Gamma \approx \Delta_2 \vee \mathrm{K}_{n-2}$ and $\deg(f_\Gamma(X)) = \binom{n}{2}$ by Proposition 7.5(ii). (Alternatively, we may combine Corollary 5.5 and Example 1.3.) ♦

8 Applications to zeta functions of graphical group schemes

We briefly relate some of our findings to recent work on zeta functions of groups.

8.1 Reminder: class-counting and conjugacy class zeta functions

The study of zeta functions associated with groups and group-theoretic counting problems goes back to influential work of Grunewald et al. [11]. Let $\mathbf{G}$ be a group scheme of finite type over a compact discrete valuation ring $\mathfrak{O}$ with maximal ideal $\mathfrak{P}$. The **class-counting zeta function** of $\mathbf{G}$ is the Dirichlet series $\zeta^{\mathrm{k}}_{\mathbf{G}}(s) = \sum_{i=0}^{\infty} \mathrm{k}(\mathbf{G}(\mathfrak{O}/\mathfrak{P}^i))|\mathfrak{O}/\mathfrak{P}^i|^{-s}$. Beginning with work of du Sautoy [7], these and closely related series enumerating conjugacy classes have recently been studied, see [3, 19, 20, 27–29]. Recall the definition of the conjugacy class zeta function $\zeta^{\mathrm{cc}}_G(s)$ associated with a finite group G from §4. Lins [19, Def. 1.2] introduced a refinement of $\zeta^{\mathrm{k}}_{\mathbf{G}}(s)$, the **bivariate conjugacy class zeta function** $\zeta^{\mathrm{cc}}_{\mathbf{G}}(s_1, s_2) = \sum_{i=0}^{\infty} \zeta^{\mathrm{cc}}_{\mathbf{G}(\mathfrak{O}/\mathfrak{P}^i)}(s_1)|\mathfrak{O}/\mathfrak{P}^i|^{-s_2}$ of $\mathbf{G}$ and studied these functions for certain classes of unipotent group schemes; note that $\zeta^{\mathrm{k}}_{\mathbf{G}}(s) = \zeta^{\mathrm{cc}}_{\mathbf{G}}(0, s)$.

Theorem 1.2 is in fact a special case of a far more general result pertaining to class-counting zeta functions associated with graphical group schemes.

Theorem 8.1 (Cf. [29, Cor. B]). *For each graph Γ, there exists a rational function $\tilde{W}_\Gamma(X,Y) \in \mathbf{Q}(X,Y)$ with the following property: for each compact discrete valuation ring $\mathfrak{O}$ with residue field size q, we have $\zeta^{\mathrm{k}}_{\mathbf{G}_\Gamma \otimes \mathfrak{O}}(s) = \tilde{W}_\Gamma(q, q^{-s})$.*

Theorem 8.1 contains Theorem 1.2 as a special case via $\tilde{W}_\Gamma(X,Y) = 1 + f_\Gamma(X)Y + \mathcal{O}(Y^2)$.

Remark 8.2. In the present article, we chose to normalise our polynomials and rational functions slightly differently compared to [29]. Namely, what we call $\tilde{W}_\Gamma(X,Y)$ here coincides with $W^-_\Gamma(X, X^m Y)$ in [29], where m is the number of edges of Γ.

8.2 Class-counting zeta functions of graphical group schemes and joins

Let Γ_1 and Γ_2 be graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Define a rational function $Q_{\Gamma_1,\Gamma_2}(X,Y) \in \mathbf{Q}(X,Y)$ via

$$\begin{aligned} Q_{\Gamma_1,\Gamma_2}(X,Y) = X^{m_1+m_2+(n_1-1)(n_2-1)}Y - 1 \\ &+ \tilde{W}_{\Gamma_1}(X, X^{m_2+(n_1-1)n_2}Y) \cdot (1 - X^{m_1+m_2+(n_1-1)n_2}Y)(1 - X^{m_1+m_2+(n_1-1)n_2+1}Y) \\ &+ \tilde{W}_{\Gamma_2}(X, X^{m_1+n_1(n_2-1)}Y) \cdot (1 - X^{m_1+m_2+n_1(n_2-1)}Y)(1 - X^{m_1+m_2+n_1(n_2-1)+1}Y). \end{aligned}$$

Our study of joins in §5 was motivated by the following.

Theorem 8.3 ([29, Prop. 8.4]). *Suppose that Γ_1 and Γ_2 are cographs. Then*

$$\tilde{W}_{\Gamma_1 \vee \Gamma_2}(X,Y) = \frac{Q_{\Gamma_1,\Gamma_2}(X,Y)}{(1 - X^{m_1+m_2+n_1n_2}Y)(1 - X^{m_1+m_2+n_1n_2+1}Y)}. \tag{8.1}$$

It remains unclear whether the assumption that Γ_1 and Γ_2 be cographs in Theorem 8.3 is truly needed or if it is merely an artefact of the proof given in [29].

Question 8.4 ([29, Question 10.1]). Does (8.1) hold for arbitrary graphs Γ_1 and Γ_2?

We obtain a positive answer to a (much weaker!) "approximate form" of Question 8.4.

Proposition 8.5. *Let Γ_1 and Γ_2 be arbitrary graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then, regarded as formal power series in Y over $\mathbf{Q}(X)$, the rational function $\tilde{W}_{\Gamma_1 \vee \Gamma_2}(X,Y)$ and the right-hand side of (8.1) agree modulo Y^2.*

Proof. This follows from Corollary 5.5: by expanding the right-hand side of (8.1) as a series in Y, we find that the coefficient of Y is given by the right-hand side of (5.1). ♦

8.3 Uniformity and the difficulty of computing zeta functions of groups

For reasons that are not truly understood at present, many interesting examples of zeta functions associated with group-theoretic counting problems are "(almost) uniform". As we now recall, the task of symbolically computing such zeta functions is well-defined.

Uniformity. Beginning with a global object G and a type of counting problem, we often obtain (a) associated local objects G_p indexed by primes (or places) p and (b) associated local zeta functions $\zeta_{G_p}(s)$. The family $(\zeta_{G_p}(s))_p$ of zeta functions is **(almost) uniform** if there exists $W_G(X,Y) \in \mathbf{Q}(X,Y)$ such that $\zeta_{G_p}(s) = W_G(p,p^{-s})$ for (almost) all p. (Stronger forms of uniformity may also take into account local base extensions or changing the characteristic of compact discrete valuation rings under consideration. Variants apply to multivariate zeta functions such as $\zeta^{\mathrm{cc}}_{\mathbf{G}}(s_1,s_2)$.) It is then natural to seek to devise algorithms for computing $W_G(X,Y)$ and to consider the complexity of such algorithms.

Numerous computations of (almost) uniform zeta functions associated with groups and related algebraic structures have been recorded in the literature; see e.g. [8]. For a recent example, Carnevale et al. [5] (see also [6]) obtained strong uniformity results for ideal zeta functions of certain nilpotent Lie rings. Their explicit formulae for rational functions as sums over chain complexes involve sums of super-exponentially many rational functions.

The following example illustrates how class-counting zeta functions associated with graphical group schemes fit the above template for uniformity of zeta functions.

Example 8.6. Let $G = \mathbf{G}_\Gamma$ be a graphical group scheme. For a prime p, let $\mathbf{Z}_p$ denote the ring of p-adic integers and let $G_p = G \otimes \mathbf{Z}_p$. Writing $\zeta_{G_p}(s) = \zeta^{\mathrm{k}}_{G_p}(s)$, the family $(\zeta_{G_p}(s))_p$ is uniform by Theorem 8.1 with $W_G(X,Y) = \tilde{W}_\Gamma(X,Y)$. The constructive proof of Theorem 8.1 in [29] gives rise to an algorithm for computing $\tilde{W}_\Gamma(X,Y)$ (see [29, §9.1]). While no complexity analysis was carried out in [29], this algorithm appears likely to be substantially worse than polynomial-time. For a cograph Γ, [29, Thms C–D] combine to produce a formula for $\tilde{W}_\Gamma(X,Y)$ as a sum of explicit rational functions, the number of which grows super-exponentially with the number of vertices of Γ.

Computing bivariate conjugacy class zeta functions. As indicated (but not spelled out as such) in [29, §8.5], Theorem 8.1 admits the following generalisation: given a graph Γ, there exists $\tilde{W}_\Gamma(X,Y,Z) \in \mathbf{Q}(X,Y,Z)$ such that for all compact discrete valuation rings with residue field size q, $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma}(s_1,s_2) = \tilde{W}_\Gamma(q,q^{-s_1},q^{-s_2})$. (Hence, $\tilde{W}_\Gamma(X,1,Z) = \tilde{W}_\Gamma(X,Z)$.) Suppose that, given Γ, an oracle provided us with $\tilde{W}_\Gamma(X,Y,Z)$ as a reduced fraction of polynomials. Since $\tilde{W}_\Gamma(X,Y,Z) = 1 + \mathsf{F}_\Gamma(X,Y)Z + \mathcal{O}(Z^2)$, we may then compute $\mathsf{F}_\Gamma(X,Y)$ by symbolic differentiation. In particular, Proposition D implies that computing $\tilde{W}_\Gamma(X,Y,Z)$ is NP-hard. To the author's knowledge, this is the first non-trivial *lower* bound for the difficulty of computing uniform zeta functions associated with groups. We do not presently obtain a similar lower bound for the difficulty of computing $\tilde{W}_\Gamma(X,Y)$ since the difficulty of determining $f_\Gamma(X)$ remained unresolved in §7.

8.4 Open problem: higher congruence levels

It is an open problem to find a combinatorial formula for the rational functions $\tilde{W}_\Gamma(X,Y)$ (or their generalisations $\tilde{W}_\Gamma(X,Y,Z)$ from §8.3) as Γ ranges over all graphs on a given vertex set; cf. [29, Question 1.8(iii)]. Corollary B provides such a formula for the first non-trivial coefficient of $\tilde{W}_\Gamma(X,Y) = 1 + f_\Gamma(X)Y + \mathcal{O}(Y^2)$, and Theorem A provides a formula for the first non-trivial coefficient of $\tilde{W}_\Gamma(X,Y,Z) = 1 + \mathsf{F}_\Gamma(X,Y)Z + \mathcal{O}(Z^2)$. As suggested by one of the anonymous referees, it is natural to ask whether a combinatorial formulae of the type considered here can be obtained for the coefficient of Y^2 in $\tilde{W}_\Gamma(X,Y)$ or of Z^2 in $\tilde{W}_\Gamma(X,Y,Z)$.

These coefficients enumerate conjugacy classes of graphical groups $\mathbf{G}_\Gamma(\mathfrak{O}/\mathfrak{P}^2)$, where $\mathfrak{O}$ is a compact discrete valuation ring with maximal ideal $\mathfrak{P}$. The "dual" problem of enumerating characters (see §1.6) is related to recent research developments. In particular, the character theory of reductive groups over rings of the form $\mathfrak{O}/\mathfrak{P}^2$ has received considerable attention; see e.g. [30, 33].

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References

[1] J. L. Arocha and B. Llano, *Mean value for the matching and dominating polynomial*, Discuss. Math. Graph Theory **20** (2000), no. 1, 57–69.

[2] P. Belkale and P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, Duke Math. J. **116** (2003), no. 1, 147–188.

[3] M. N. Berman, J. Derakhshan, U. Onn, and P. Paajanen, *Uniform cell decomposition with applications to Chevalley groups*, J. Lond. Math. Soc. (2) **87** (2013), no. 2, 586–606.

[4] N. Biggs, *Algebraic graph theory*, Second, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993.

[5] A. Carnevale, M. M. Schein, and C. Voll, *Generalized Igusa functions and ideal growth in nilpotent Lie rings (preprint)* (2019). arXiv:1903.03090.

[6] A. Carnevale, M. M. Schein, and C. Voll, *Generalized Igusa functions and ideal growth in nilpotent Lie rings*, Sém. Lothar. Combin. **84B** (2020), Art. 71, 12.

[7] M. P. F. du Sautoy, *Counting conjugacy classes*, Bull. London Math. Soc. **37** (2005), no. 1, 37–44.

[8] M. P. F. du Sautoy and L. Woodward, *Zeta functions of groups and rings*, Lecture Notes in Mathematics, vol. 1925, Springer-Verlag, Berlin, 2008.

[9] M. R. Garey and D. S. Johnson, *Computers and intractability*, W. H. Freeman and Co., San Francisco, Calif., 1979. A guide to the theory of NP-completeness.

[10] S. M. Goodwin and G. Röhrle, *Calculating conjugacy classes in Sylow p-subgroups of finite Chevalley groups*, J. Algebra **321** (2009), no. 11, 3321–3334.

[11] F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), no. 1, 185–223.

[12] P. Hansen, *Upper bounds for the stability number of a graph*, Rev. Roumaine Math. Pures Appl. **24** (1979), no. 8, 1195–1199.

[13] X. He and Y. Qiao, *On the Baer-Lovász-Tutte construction of groups from graphs: isomorphism types and homomorphism notions*, European J. Combin. **98** (2021), Paper No. 103404, 12.

[14] G. Higman, *Enumerating p-groups. I. Inequalities*, Proc. London Math. Soc. (3) **10** (1960), 24–30.

[15] B. Huppert, *Endliche Gruppen I*, Die Grundlehren der mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967.

[16] N. Ito and A. Mann, *Counting classes and characters of groups of prime exponent*, Israel J. Math. **156** (2006), 205–220.

[17] Y. Li and Y. Qiao, *Group-theoretic generalisations of vertex and edge connectivities*, Proc. Amer. Math. Soc. **148** (2020), no. 11, 4679–4693.

[18] M. W. Liebeck and A. Shalev, *Character degrees and random walks in finite groups of Lie type*, Proc. London Math. Soc. (3) **90** (2005), no. 1, 61–86.

[19] P. M. Lins de Araujo, *Bivariate representation and conjugacy class zeta functions associated to unipotent group schemes, I: Arithmetic properties*, J. Group Theory **22** (2019), no. 4, 741–774.

[20] P. M. Lins de Araujo, *Bivariate representation and conjugacy class zeta functions associated to unipotent group schemes, II: groups of type F, G, and H*, Internat. J. Algebra Comput. **30** (2020), no. 5, 931–975.

[21] M. Marjoram, *Irreducible characters of small degree of the unitriangular group*, Irish Math. Soc. Bull. **42** (1999), 21–31.

[22] A. H. Mekler, *Stability of nilpotent groups of class* 2 *and prime exponent*, J. Symbolic Logic **46** (1981), no. 4, 781–788.

[23] D. A. Mojdeh and A. S. Emadi, *Connected domination polynomial of graphs*, Fasc. Math. **60** (2018), 103–121.

[24] E. A. O'Brien and C. Voll, *Enumerating classes and characters of p-groups*, Trans. Amer. Math. Soc. **367** (2015), no. 11.

[25] I. Pak and A. Soffer, *On Higman's $k(U_n(\mathbb{F}_q))$ conjecture (preprint)* (2015). arXiv:1507.00411.

[26] P. Renteln, *The Hilbert series of the face ring of a flag complex*, Graphs Combin. **18** (2002), no. 3, 605–619.

[27] T. Rossmann, *The average size of the kernel of a matrix and orbits of linear groups*, Proc. Lond. Math. Soc. (3) **117** (2018), no. 3, 574–616.

[28] T. Rossmann, *The average size of the kernel of a matrix and orbits of linear groups, II: duality*, J. Pure Appl. Algebra **224** (2020), no. 4, 28 pages.

[29] T. Rossmann and C. Voll, *Groups, graphs, and hypergraphs: average sizes of kernels of generic matrices with support constraints*, To appear in Mem. Amer. Math. Soc. (2021^+). arXiv:1908.09589.

[30] P. Singla, *On representations of general linear groups over principal ideal local rings of length two*, J. Algebra **324** (2010), no. 9, 2543–2563.

[31] R. P. Stanley, *Enumerative combinatorics. Volume 1*, Second edition, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.

[32] A. Stasinski and C. Voll, *Representation zeta functions of nilpotent groups and generating functions for Weyl groups of type B*, Amer. J. Math. **136** (2014), no. 2, 501–550.

[33] A. Stasinski and A. Vera-Gajardo, *Representations of reductive groups over finite local rings of length two*, J. Algebra **525** (2019), 171–190.

[34] A. Vera-López and J. M. Arregi, *Polynomial properties in unitriangular matrices*, J. Algebra **244** (2001), no. 1, 343–351.

[35] A. Vera-López and J. M. Arregi, *Conjugacy classes in unitriangular matrices*, Linear Algebra Appl. **370** (2003), 85–124.

[36] A. Vera-López and J. M. Arregi, *Conjugacy classes in Sylow p-subgroups of* $\mathrm{GL}(n,q)$, J. Algebra **152** (1992), no. 1, 1–19.

[37] C. Q. Zhang, *Finding critical independent sets and critical vertex subsets are polynomial problems*, SIAM J. Discrete Math. **3** (1990), no. 3, 431–438.

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